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The Phase Transition of the Quantum Ising Model is Sharp

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Abstract An analysis is presented of the phase transition of the quantum Ising model with transverse field on the d-dimensional hypercubic lattice. It is shown that there is a unique sharp transition. The value of the critical point is calculated rigorously in one dimension. The first step is to express the quantum Ising model in terms of a (continuous) classical Ising model in $d+1$ dimensions. A so-called 'random-parity' representation is developed for the latter model, similar to the random-current representation for the classical Ising model on a discrete lattice. Certain differential inequalities are proved. Integration of these inequalities yields the sharpness of the phase transition, and also a number of other facts concerning the critical and near-critical behaviour of the model under study.

Keywords Quantum Ising model · Ising model · Random-parity representation · Random-current representation · Random-cluster model · Differential inequality · Phase transition

1 Introduction

Geometric or 'graphical' methods have been very useful in the rigorous study of lattice models in classical statistical mechanics. Of the many examples, we mention the use of the random-cluster (or 'FK') representation to prove the existence of non-translation-invariant 'Dobrushin' states in the q-state Potts model [20]; the use of the related 'loop' representation to prove conformal invariance for the two-dimensional Ising model [36]; the use of the random-current representation to prove the sharpness of the phase transition in classical Ising models [7]. In contrast, graphical methods for *quantum* lattice models have received

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less attention. We shall formulate a so-called 'random-parity representation' for the quantum Ising model on a graph G (or, more precisely, for the corresponding 'continuous Ising model' on $G \times \mathbb{R}$, [5, 9]), and shall use it to prove the sharpness of the phase transition for this model in a general number of dimensions. The random-parity representation is a cousin of the random-current representation in [1, 7].

Let $L = (V, E)$ be a finite graph. The Hamiltonian of the quantum Ising model with transverse field on L is the matrix (or 'operator')

$$H = -\frac{1}{2}\lambda \sum_{e=uv\in E} \sigma_u^{(3)}\sigma_v^{(3)} - \delta \sum_{v\in V} \sigma_v^{(1)}, \tag{1.1}$$

acting on the Hilbert space $\mathcal{H} = \bigotimes_{v\in V} \mathbb{C}^2$. Here, the Pauli spin-$\frac{1}{2}$ matrices are given as

$$\sigma_v^{(3)} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \sigma_v^{(1)} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{1.2}$$

The constants $\lambda, \delta > 0$ in (1.1) are the spin-coupling and transverse-field intensities, respectively. The basic operator of the quantum Ising model is $e^{-\beta H}$ where $\beta > 0$. The model was introduced in [32], and has been widely studied since. See, for example, the references in [27].

It is standard (see [5, 9] for example) that the quantum Ising model on L possesses a type of 'path integral representation', which expresses it as a type of classical Ising model (or equivalently as a continuum random-cluster model with $q = 2$) on the continuous space $V \times [0, \beta]$. This representation permits the use of geometrical methods in studying the behaviour of the original quantum model. In particular, it is a useful way of establishing the existence of the infinite-volume limits as $\beta \to \infty$ and $|V| \to \infty$, and of relating the phase transition of the quantum model to that of the continuous classical model.

The main technique of this article is a type of random-current representation, called the 'random-parity' representation, for the Ising model on $V \times \mathbb{R}$. This enables a detailed analysis of the phase transition of the latter model, and hence of the related quantum model. Further details and references will be provided in the next section.

The quantum model is said to be in the 'ground state' when the limit $\beta \to \infty$ is taken. The value of β appears in the superscript of quantities that follow; when the superscript is ∞, this is to be interpreted as the relevant ground-state quantity.

Our main choice for L is a box in the d-dimensional cubic lattice $\mathbb{Z}^d$ where $d \geq 1$, with a periodic boundary condition, and we shall pass to the infinite-volume limit as $L \uparrow \mathbb{Z}^d$. (Similar results hold for other lattices, and for summable translation-invariant interactions.) The model is over-parametrized. We shall normally assume $\delta = 1$, and write $\rho = \lambda/\delta$, while noting that the same analysis holds for $\delta \in (0, \infty)$. As remarked above, one may study the quantum phase transition via that of the Ising model on the continuum $\mathbb{Z}^d \times [0, \beta]$ and, in the latter case, one may introduce the notions of magnetization $M = M^\beta(\rho, \gamma)$ and (magnetic) susceptibility $\chi = \chi^\beta(\rho, \gamma)$, where γ denotes external field. The *critical point* $\rho_c = \rho_c^\beta$ is given by

$$\rho_c^\beta := \inf\{\rho : M_+^\beta(\rho) > 0\}, \tag{1.3}$$

where

$$M_+^\beta(\rho) := \lim_{\gamma\downarrow 0} M^\beta(\rho, \gamma), \tag{1.4}$$

is the magnetization in the limiting state $\langle\cdot\rangle_+^\beta$ as $\gamma \downarrow 0$. It may be proved by standard methods that:

$$\begin{aligned} &\text{if } d \ge 2: \quad 0 < \rho_c^\beta < \infty \quad \text{ for } \beta \in (0, \infty], \\ &\text{if } d = 1: \quad \rho_c^\beta = \infty \quad \text{ for } \beta \in (0, \infty),\ 0 < \rho_c^\infty < \infty. \end{aligned} \tag{1.5}$$

When $\beta < \infty$, the magnetization, susceptibility, and critical values depend also on the parameter λ, but we suppress this for brevity of notation.

Complete statements of our main results are deferred until Sects. 6 and 7. Here are two examples of what can be proved.

Theorem 1.1 *Let $u, v \in \mathbb{Z}^d$ where $d \ge 1$, and $s, t \in \mathbb{R}$. For $\beta \in (0, \infty]$:*

(i) *if $0 < \rho < \rho_c^\beta$, the two-point correlation function $\langle\sigma_{(u,s)}\sigma_{(v,t)}\rangle_+^\beta$ of the Ising model on $\mathbb{Z}^d \times \mathbb{R}$ decays exponentially to 0 as $|u - v| + |s - t| \to \infty$,*
(ii) *if $\rho > \rho_c^\beta$, $\langle\sigma_{(u,s)}\sigma_{(v,t)}\rangle_+^\beta \ge M_+^\beta(\rho)^2 > 0$.*

Theorem 1.2 *Let $\beta \in (0, \infty]$. In the notation of Theorem 1.1, there exists $c = c(d) > 0$ such that*

$$M_+^\beta(\rho) \ge c(\rho - \rho_c^\beta)^{1/2} \quad \text{for } \rho > \rho_c^\beta.$$

These and other facts will be stated and proved in Sect. 6. Their implications for the infinite-volume quantum model will be elaborated in the next section, see in particular (2.12)–(2.14). Roughly speaking, they imply that the two-point function of the quantum model decays exponentially when $\rho < \rho_c^\beta$, and is uniformly bounded below by $c(\rho - \rho_c^\beta)$ when $\rho > \rho_c^\beta$.

The approach used here is to prove a family of differential inequalities for the magnetization $M^\beta(\rho, \gamma)$. This parallels the methods established in [4, 7] for the analysis of the phase transitions in percolation and Ising models on discrete lattices, and indeed our arguments are closely related to those of [7]. Whereas new problems arise in the current context and require treatment, certain aspects of the analysis presented here are simpler than the corresponding steps of [7]. The application to the quantum model imposes a periodic boundary condition in the β direction; the same conclusions are valid for the space–time Ising model with a free boundary condition.

The critical value ρ_c^β depends of course on the number of dimensions. We shall use planar duality to show that $\rho_c^\infty = 2$ when $d = 1$, and in addition that the transition is of second order in that $M_+^\infty(2) = 0$. See Theorem 7.1. The one-dimensional critical point has been calculated by other means in the quantum case, but we believe that the current proof is valuable. Two applications to the work of [11, 27] are summarized in Sect. 7.

Here is a brief outline of the contents of this article. Formal definitions are presented in Sect. 2. The random-parity representation of the quantum Ising model is described in Sect. 3. This representation may at first sight seem quite different from the random-current representation of the classical Ising model on a discrete lattice. It requires more work to set up than does its discrete cousin, but once in place it works in a very similar, and sometimes simpler, manner. We then state and prove, in Sect. 4.1, the fundamental 'switching lemma'. In Sect. 4.2 are presented a number of important consequences of the switching lemma, including GHS and Simon–Lieb inequalities, as well as other useful inequalities and identities. In Sect. 5, we prove the somewhat more involved differential inequality of the forthcoming Theorem 2.2, which is similar to the main inequality of [7]. Our main results follow from

Theorem 2.2 in conjunction with the results of Sect. 4.2. Finally, in Sects. 6 and 7, we give rigorous formulations and proofs of our main results.

We mention that the continuous Ising model possesses a representation of random-cluster-type; see, for example, [9, 25, 27]. This is convenient for proving various facts including the existence of infinite-volume limits. Only occasional use is made of the random-cluster representation here, and full details are omitted. See Remark 6.1.

Remark 1.3 There is a very substantial overlap between the results reported here and those of the independent and contemporaneous article [17]. The basic differential inequalities of Theorems 2.2 and 4.10 appear in both articles. The proofs are in essence the same despite some differences of presentation. We are grateful to the authors of [17] for explaining the relationship between the random-parity representation of Sect. 3 and the random-current representation of [28, Sect. 2.2]. As pointed out in [17], the appendix of [16] contains a type of switching argument for the mean-field model. A principal difference between that argument and those of [17, 28] and the current work is that it uses the classical switching lemma developed in [1], applied to a discretized version of the mean-field system.

2 Classical and Quantum Ising Models

Let $L = (V, E)$ be a finite, connected graph, which (for simplicity only) we assume to have neither loops nor multiple edges. An edge of L with endpoints u, v is denoted by uv. We write $u \sim v$ if $uv \in E$.

2.1 Quantum Ising Model with Transverse Field

As basis for each copy of $\mathbb{C}^2$ in the Hilbert space $\mathcal{H} = \bigotimes_{x\in V} \mathbb{C}^2$, we take the vectors $|+_v\rangle = \binom{1}{0}$ and $|-_v\rangle = \binom{0}{1}$. Let D be the set of $2^{|V|}$ basis vectors of $\mathcal{H}$ of the form $|\sigma\rangle = \bigotimes_{v\in V} |\pm_v\rangle$. There is a natural one–one correspondence between D and the space $\Sigma = \{-1, +1\}^V$, and we shall speak of $\mathcal{H}$ as being generated by Σ. The *trace* of the Hermitian matrix A is defined as

$$\operatorname{tr}(A) = \sum_{\sigma\in\Sigma} \langle\sigma|A|\sigma\rangle.$$

Here, $\langle\psi|$ is the adjoint, or complex transpose, of the vector $|\psi\rangle$.

The Hamiltonian of the quantum Ising model with transverse field is given in (1.1). Let $\beta > 0$ be a fixed real number (known as the 'inverse temperature'), and define the positive temperature states

$$\nu_{L,\beta}(Q) = \frac{1}{Z_L(\beta)} \operatorname{tr}(e^{-\beta H} Q), \tag{2.1}$$

where $Z_L(\beta) = \operatorname{tr}(e^{-\beta H})$ and Q is a suitable matrix. The *ground state* is defined as the limit ν_L of $\nu_{L,\beta}$ as $\beta \to \infty$. If $(L_n : n \geq 1)$ is an increasing sequence of graphs tending to an infinite graph L, then we may also make use of the *infinite-volume* limits

$$\nu_{L,\beta} = \lim_{n\to\infty} \nu_{L_n,\beta}, \qquad \nu_L = \lim_{n\to\infty} \nu_{L_n}.$$

The existence of such limits is discussed in [9].

2.2 Space–Time Ising Model

A number of authors have developed and utilized the following 'path integral representation' of the quantum Ising model, see for example [5, 9, 15, 16, 27, 33] and the recent surveys to be found in [24, 28]. Let $\mathbb{S} = \mathbb{S}_\beta$ be the circle of circumference β, which we think of as the interval $[0, \beta]$ with its two endpoints identified. Let λ, δ, γ be non-negative constants, and let μ_λ, μ_δ, μ_γ be the probability measures associated with independent Poisson processes on $E \times \mathbb{S}$, $V \times \mathbb{S}$, and $V \times \mathbb{S}$ with respective intensities λ, δ, γ. Elements sampled from these measures will typically be denoted by B, D, G, and their members will be called *bridges*, *deaths* and *ghost-bonds* respectively.

Remark 2.1 For simplicity of notation in this article, we shall frequently overlook events with zero probability.

Thus, for example, we shall assume without more ado that the $\mathbb{S}$ coordinates of the points of $B \cup D \cup G$ are distinct. Furthermore, we shall take as sample space for B (respectively, D, G) the set $\mathcal{B}$ (respectively, $\mathcal{F}$) of finite subsets of $E \times \mathbb{S}$ (respectively, $V \times \mathbb{S}$).

For $D \in \mathcal{F}$, write $V(D)$ for the collection of maximal intervals of $(V \times \mathbb{S}) \setminus D$, and let $\Sigma(D) = \{-1, +1\}^{V(D)}$. Each $\sigma \in \Sigma(D)$ should be viewed as a spin-configuration on $(V \times \mathbb{S}) \setminus D$ using local spins ± 1: for $x = (v, t) \in (V \times \mathbb{S}) \setminus D$, write $\sigma_x = \sigma_{(v,t)}$ for the local state of x under σ, that is, the σ-value of the interval in $V(D)$ containing x. Note that σ_x is undefined for $x \notin D$, but, since D is almost surely finite, this is no bar to the following definition.

The *space–time Ising measure* on the domain

$$\Lambda := L \times \mathbb{S} = (V \times \mathbb{S}, E \times \mathbb{S})$$

is defined to be the probability measure on the space

$$\Sigma = \bigcup_{D \in \mathcal{F}} \Sigma(D),$$

with partition function

$$Z' = \int_{\mathcal{F}} d\mu_\delta(D) \sum_{\sigma \in \Sigma(D)} \exp\left\{ \lambda \int_{E \times \mathbb{S}} \sigma_e \, de + \gamma \int_{V \times \mathbb{S}} \sigma_x \, dx \right\} \tag{2.2}$$

where $\sigma_e = \sigma_{(u,t)} \sigma_{(v,t)}$ if $e = (uv, t)$. The two integrals in (2.2) are to be interpreted, respectively, as

$$\sum_{e = uv \in E} \int_{\mathbb{S}} \sigma_{(u,t)} \sigma_{(v,t)} \, dt, \qquad \sum_{v \in V} \int_{\mathbb{S}} \sigma_{(v,t)} \, dt.$$

Note that the use of the *circle* $\mathbb{S}$ amounts to a periodic boundary condition in the β direction. We shall generally suppress reference to β in the following.

Here is a word of motivation for (2.2); see also [12, 27]. Let $D \in \mathcal{F}$, and think of $V(D)$ as the set of vertices of a graph with edges given as follows. We augment $V(D)$ with an auxiliary vertex, called the *ghost-vertex* and denoted Γ, to which we assign spin $\sigma_\Gamma = 1$. An edge is placed between Γ and each $\bar{v} \in V(D)$. For $\bar{u}, \bar{v} \in V(D)$, with $\bar{u} = u \times I_1$ and $\bar{v} = v \times I_2$ say, we place an edge between $\bar{u}$ and $\bar{v}$ if and only if: (i) uv is an edge of L, and (ii) $I_1 \cap I_2 \neq \varnothing$. Under the measure with partition function (2.2), and conditional on D, a spin-configuration $\sigma \in \Sigma(D)$ on this graph receives an Ising weight

$$\exp\left\{\sum_{\bar{u}\bar{v}} J_{\bar{u}\bar{v}}\sigma_{\bar{u}}\sigma_{\bar{v}} + \sum_{\bar{v}} h_{\bar{v}}\sigma_{\bar{v}}\right\}, \tag{2.3}$$

where $\sigma_{\bar{v}}$ denotes the common value of σ along $\bar{v}$, and with $J_{\bar{u}\bar{v}} = \lambda|I_1 \cap I_2|$ and $h_{\bar{v}} = \gamma|\bar{v}|$. Here, $|J|$ denotes the Lebesgue measure of the interval J. This observation will be pursued further in Sect. 3.2.

We will use angle brackets $\langle\cdot\rangle$ for the expectation operator under the measure given by (2.2). Thus, for example,

$$\langle\sigma_A\rangle = \frac{1}{Z'}\int d\mu_\delta(D) \sum_{\sigma\in\Sigma(D)} \sigma_A \exp\left\{\lambda\int_{E\times\mathbb{S}} \sigma_e\, de + \gamma\int_{V\times\mathbb{S}} \sigma_x\, dx\right\}, \tag{2.4}$$

where $A \subseteq V \times \mathbb{S}$ is a finite set, and

$$\sigma_A := \prod_{y\in A} \sigma_y. \tag{2.5}$$

Let 0 be a given point of $V \times \mathbb{S}$. We will be particularly concerned with the *magnetization* and *susceptibility* of the space–time Ising model on $\Lambda = L \times \mathbb{S}$, given respectively by

$$M = M_\Lambda(\lambda,\delta,\gamma) := \langle\sigma_0\rangle, \tag{2.6}$$

$$\chi = \chi_\Lambda(\lambda,\delta,\gamma) := \frac{\partial M}{\partial\gamma} = \int_\Lambda \langle\sigma_0;\sigma_x\rangle\, dx, \tag{2.7}$$

where the *truncated* two-point function $\langle\sigma_0;\sigma_x\rangle$ is given by

$$\langle\sigma_A;\sigma_B\rangle := \langle\sigma_A\sigma_B\rangle - \langle\sigma_A\rangle\langle\sigma_B\rangle. \tag{2.8}$$

We will derive a number of differential inequalities for M and χ, of which the following is the principal one. In writing $L = [-n,n]^d$, we mean that L is the box $[-n,n]^d$ of $\mathbb{Z}^d$ with 'periodic boundary conditions', which is to say that two vertices u, v are joined by an edge whenever there exists $i \in \{1,2,\dots,d\}$ such that: u and v differ by exactly $2n$ in the ith coordinate, and the other coordinates are equal. (Our results are in fact valid in greater generality, see the statement before Assumption 4.9.) Subject to this boundary condition, M and χ do not depend on the choice of origin 0.

Theorem 2.2 *Let $d \geq 1$ and let $L = [-n,n]^d$. Then*

$$M \leq \gamma\chi + M^3 + 2\lambda M^2\frac{\partial M}{\partial\lambda} - 2\delta M^2\frac{\partial M}{\partial\delta}. \tag{2.9}$$

A similar inequality was derived in [7] for the classical Ising model, and our method of proof is closely related to that used there. Other such inequalities have been proved for percolation in [4] (see also [22]), and for the contact model in [6, 10]. As observed in [4, 7], the powers of M on the right side of (2.9) determine the bounds of Theorems 1.1(ii) and 1.2 on the critical exponents. The cornerstone of our proof is a random-parity representation of the space–time Ising model.

In the ground-state limit as $\beta, n \to \infty$ and $\gamma > 0$, the two quantities M, χ have well-defined limits denoted M_∞ and χ_∞. By a re-scaling argument, M_∞ depends on the parameters through the ratios λ/δ, γ/δ. Thus we may take as 'order parameter' the function

$$M(\rho, \gamma) := M_\infty(\rho, 1, \gamma).$$

More generally, let $M^\beta(\rho, \gamma) = M_\infty^\beta(\rho, 1, \gamma)$ where $M_\infty^\beta = \lim_{n\to\infty} M^\beta$, and define the critical value ρ_c^β by (1.3).

The analysis of the differential inequalities, following [4, 7], reveals a number of facts about the behaviour of the model. In particular, we will show the exponential decay of the correlations $\langle \sigma_0 \sigma_x \rangle_+^\beta$ when $\rho < \rho_c^\beta$ and $\gamma = 0$, as asserted in Theorem 1.1, and in addition certain bounds on two critical exponents of the model. See Sect. 6 for further details.

We shall on occasion write $\mu(f)$ for the expectation of a random variable f under the probability measure μ. The indicator function of an event H is written either 1_H or $1\{H\}$. The complement of H is written H^c.

2.3 Classical/Quantum Relationship

The space–time Ising model is closely related to the quantum Ising model, one manifestation of this being the following. As indicated at the start of this section, a classical spin configuration $\sigma \in \Sigma = \{-1, +1\}^V$ may be identified with the basis vector $|\sigma\rangle = \bigotimes_{v\in V} |\sigma_v\rangle$ of $\mathcal{H}$. The state $\nu_{L,\beta}$ of (2.1) gives rise thereby to a probability measure μ on Σ by

$$\mu(\sigma) = \frac{\langle \sigma | e^{-\beta H} | \sigma \rangle}{\operatorname{tr}(e^{-\beta H})}, \quad \sigma \in \Sigma. \tag{2.10}$$

When $\gamma = 0$, it turns out that μ is the law of the vector $(\sigma_{(v,0)} : v \in V)$ under the space–time Ising measure of (2.2) (see [9] and the references therein). It therefore makes sense to study the phase diagram of the quantum Ising model via its representation in the space–time Ising model. Note, however, that in our analysis it is crucial to work with $\gamma > 0$, and to take the limit $\gamma \downarrow 0$ later. The role played in the classical model by the external field will in our analysis be played by the 'ghost-field' γ rather than the 'physical' transverse field δ.

We draw from [5, 9] in the following summary of the relationship between the phase transitions of the quantum and space–time Ising models. Let $u, v \in V$, and

$$\tau_L^\beta(u, v) := \operatorname{tr}\big(\nu_{L,\beta}(Q_{u,v})\big), \quad Q_{u,v} = \sigma_u^{(3)} \sigma_v^{(3)}.$$

It is the case that

$$\tau_L^\beta(u, v) = \langle \sigma_A \rangle_L^\beta \tag{2.11}$$

where $A = \{(u, 0), (v, 0)\}$, and the role of β is emphasized in the superscript. Let τ_L^∞ denote the limit of τ_L^β as $\beta \to \infty$. For $\beta \in (0, \infty]$, let τ^β be the limit of τ_L^β as $L \uparrow \mathbb{Z}^d$. (The existence of this limit may depend on the choice of boundary condition on L, and we return to this at the end of Sect. 6.) By Theorem 1.1,

$$\tau^\beta(u, v) \le c' e^{-c|u-v|}, \tag{2.12}$$

where c', c depend on ρ, and $c > 0$ for $\rho < \rho_c^\beta$ and $\beta \in (0, \infty]$. Here, $|u - v|$ denotes the L^1 distance from u to v. The situation when $\rho = \rho_c^\beta$ is more obscure, but one has that

$$\limsup_{|v|\to\infty} \tau^\beta(u, v) \le M_+^\beta(\rho), \tag{2.13}$$

so that $\tau^\beta(u,v) \to 0$ as $|v| \to \infty$, whenever $M_+^\beta(\rho) = 0$. It is proved at Theorem 7.1 that $\rho_c^\infty = 2$ and $M_+^\infty(2) = 0$ when $d = 1$.

By the FKG inequality, and the uniqueness of infinite clusters in the continuum random-cluster model (see [9, 25], for example),

$$\tau^\beta(u,v) \geq M_+^\beta(\rho-)^2 > 0, \tag{2.14}$$

when $\rho > \rho_c^\beta$ and $\beta \in (0,\infty]$, where $f(x-) := \lim_{y\uparrow x} f(y)$. The proof is discussed at the end of Sect. 6.

The quantum mean-field, or Curie–Weiss, model has been studied using large-deviation techniques in [16], see also [25]. A random-current representation of the quantum Ising model may be found in [28], and, as explained in Remark 1.3 and [17], this is intimately related to that discussed and exploited in the next section.

3 The Random-Parity Representation

The Ising model on a discrete graph L is a 'site model', in the sense that configurations comprise spins assigned to the vertices (or 'sites') of L. The classical random-current representation maps this into a bond-model, in which the sites no longer carry random values, but instead the *edges* e (or 'bonds') of the graph are replaced by a random number N_e of parallel edges. The bond e is called *even* (respectively, *odd*) if N_e is even (respectively, odd). The odd bonds may be arranged into paths and cycles. One cannot proceed in the same way in the above space–time Ising model.

There are two possible alternative approaches. The first uses the fact that, conditional on the set D of deaths, Λ may be viewed as a discrete structure with finitely many components, to which the random-current representation of [1] may be applied; this is explained in detail around (3.16) below. Another approach is to forget about 'bonds', and instead to concentrate on the parity configuration associated with a current-configuration, as follows. The relationship with the random-current representation of [28] is discussed in Remark 1.3.

The circle $\mathbb{S}$ may be viewed as a continuous limit of a ring of equally spaced points. If we apply the random-current representation to the discretized system, but only record whether a bond is even or odd, the representation has a well-defined limit as a partition of $\mathbb{S}$ into even and odd sub-intervals. In the limiting picture, even and odd intervals carry different weights, and it is the properties of these weights that render the representation useful. This is the essence of the main result in this section, Theorem 3.1. We will prove this result without recourse to discretization.

3.1 Colourings

We first generalize the set-up of Sect. 2. For $v \in V$, let $K_v \subseteq \mathbb{S}$ be a finite union of (maximal) disjoint intervals, say $K_v = \bigcup_{i=1}^{m(v)} I_i^v$. No assumption is made at this stage on whether the I_i^v are open, closed, or half-open. For $e = uv \in E$, let $K_e = K_u \cap K_v$. With the K_v given, we define

$$K := \bigcup_{v\in V} v \times K_v, \qquad F := \bigcup_{e\in E} e \times K_e, \tag{3.1}$$

$$\Lambda := (K, F), \tag{3.2}$$

where these sets are considered as unions of real intervals. We shall soon introduce an auxiliary 'ghost-vertex', denoted Γ, and shall write

$$K^{\Gamma} := K \cup \{\Gamma\}. \tag{3.3}$$

In Sect. 2, we treated only the case when each K_v comprises the single interval $\mathbb{S} := [0, \beta]$. We continue to use the notation $\mathcal{B}$ (respectively, $\mathcal{F}$) for the set of finite subsets of F (respectively, K). The closure of a Borel subset J of $\mathbb{Z} \times \mathbb{R}$ is written $\overline{J}$.

Much of the following analysis is valid with the constants λ, δ, γ replaced by (possibly non-constant) functions. Specifically, let $\lambda : E \times \mathbb{S} \to \mathbb{R}_+$, $\delta : V \times \mathbb{S} \to \mathbb{R}_+$, and $\gamma : V \times \mathbb{S} \to \mathbb{R}_+$ be bounded, measurable functions, where $\mathbb{R}_+ = [0, \infty)$. We retain the notation λ, δ, γ for the restrictions of these functions to Λ, given in (3.2), and let μ_λ, μ_δ, μ_γ be the probability measures associated with independent Poisson processes with respective intensities λ, δ, γ on the respective subsets of Λ. For $D \in \mathcal{F}$, the set $(v \times K_v) \setminus D$ is a union of maximal death-free intervals $v \times J_v^k$, where $k = 1, 2, \dots, n$ and $n = n(v, D)$ is the number of such intervals. With $V(D)$ the collection of all such intervals, and $\Sigma(D) = \{-1, +1\}^{V(D)}$ as before, we may define the space–time Ising measure on the Λ of (3.2) as that with partition function

$$Z'_K = \int_{\mathcal{F}} d\mu_\delta(D) \sum_{\sigma \in \Sigma(D)} \exp\left\{ \int \lambda(e)\sigma_e \, de + \int \gamma(x)\sigma_x \, dx \right\}. \tag{3.4}$$

As in (2.4), we write $\langle \sigma_A \rangle_K$, abbreviated to $\langle \sigma_A \rangle$ when the context is obvious, for the mean of σ_A under this measure.

It is essential for our method that we work on general domains of the form given in (3.2). The reason for this is that, in the geometrical analysis of currents, we shall at times remove from K a random subset called the 'backbone', and the ensuing domain has the form of (3.2). This generalization also allows us to work with a 'free' rather than a 'vertically periodic' boundary condition. That is, by setting $K_v = [0, \beta)$ for all $v \in V$, rather than $K_v = [0, \beta]$, we effectively remove the restriction that the 'top' and 'bottom' of each $v \times \mathbb{S}$ have the same spin.

Whenever we wish to emphasize the roles of particular K, λ, δ, γ, we include them as subscripts. For example, we may write $\langle \sigma_A \rangle_K$ or $\langle \sigma_A \rangle_{K,\gamma}$ or Z'_γ, and so on.

We now define two additional random processes associated with the space–time Ising measure on Λ. The first is a random colouring of K, and the second is a random (finite) weighted graph. These two objects will be the main components of the random-parity representation.

Let $\overline{K}$ be the closure of K. A set of *sources* is a finite set $A \subseteq \overline{K}$ such that: each $a \in A$ is the endpoint of at most one maximal sub-interval I_i^v of K. (This last condition is for simplicity later.) Let $B \in \mathcal{B}$ and $G \in \mathcal{F}$. Let $S = A \cup G \cup V(B)$, where $V(B)$ is the set of endpoints of bridges of B, and call members of S *switching points*. As in Remark 2.1, we shall assume that A, G and $V(B)$ are disjoint.

We shall define a colouring $\psi^A = \psi^A(B, G)$ of $K \setminus S$ using the two colours (or labels) 'even' and 'odd'. This colouring is constrained to be 'valid', where a valid colouring is defined to be a mapping $\psi : K \setminus S \to \{\text{even}, \text{odd}\}$ such that:

(i) the label is constant between two neighbouring switching points, that is, ψ is constant on any sub-interval of K containing no members of S,
(ii) the label always switches at each switching point, which is to say that, for $(u, t) \in S$, $\psi(u, t-) \neq \psi(u, t+)$, whenever these two values are defined,

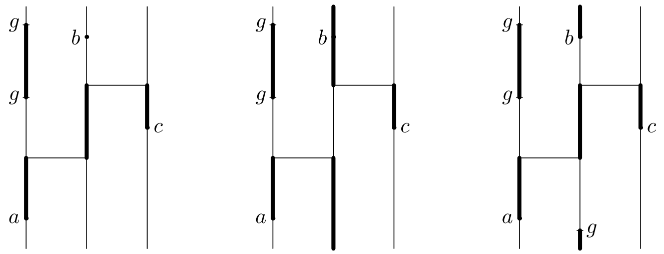

Fig. 1 Three examples of colourings for given $B \in \mathcal{B}$, $G \in \mathcal{F}$. Points in G are written g. *Thick line* segments are 'odd' and thin segments 'even'. In this illustration we have taken $K_v = \mathbb{S}$ for all v. *Left and middle*: two of the eight possible colourings when the sources are a, c. *Right*: one of the possible colourings when the sources are a, b, c

(iii) for any pair v, k such that $I_k^v \neq \mathbb{S}$, in the limit as we move along $v \times I_k^v$ towards an endpoint a of $v \times I_k^v$, the colour converges to 'even' if $a \notin A$, and to 'odd' if $a \in A$.

If there exists $v \in V$ and $1 \leq k \leq m(v)$ such that $v \times \overline{I_k^v}$ contains an *odd* number of switching points, then conditions (i)–(iii) cannot be satisfied; in this case we set the colouring ψ^A to a default value denoted #.

Suppose that (i)–(iii) *can* be satisfied, and let

$$W = W(K) := \{v \in V : K_v = \mathbb{S}\}.$$

If $W = \varnothing$ (in which case we speak of a 'free' boundary condition), then there exists a unique valid colouring, denoted ψ^A. If $r = |W| \geq 1$, there are exactly 2^r valid colourings, one for each of the two possible colours assignable to the sites $(w, 0)$, $w \in W$; in this case we let ψ^A be chosen uniformly at random from this set, independently of all other choices. (If $(w, 0) \in S$, we work instead with the colour of (w, ε) in the limit as $\varepsilon \downarrow 0$.)

Let $M_{B,G}$ be the probability measure (or expectation when appropriate) governing the randomization in the definition of ψ^A: $M_{B,G}$ is the uniform (product) measure on the set of valid colourings, and it is a point mass if and only if $W = \varnothing$. See Fig. 1.

Fix the set A of sources. For (almost every) pair B, G, one may construct as above a (possibly random) colouring ψ^A. Conversely, it is easily seen that the pair B, G may (almost surely) be reconstructed from knowledge of the colouring ψ^A. For given A, we may thus speak of a configuration as being either a pair B, G, or a colouring ψ^A. While $\psi^A(B, G)$ is a colouring of $K \setminus S$ only, we shall sometimes refer to it as a colouring of K.

The next step is to assign weights $\partial\psi$ to colourings ψ. The 'failed' colouring # is assigned weight $\partial\# = 0$. For every valid colouring ψ, let $\mathrm{ev}(\psi)$ (respectively, $\mathrm{odd}(\psi)$) denote the subset of K that is labelled even (respectively, odd), and let

$$\partial\psi := \exp\{2\delta(\mathrm{ev}(\psi))\}, \tag{3.5}$$

where

$$\delta(U) := \int_U \delta(x)\, dx, \quad U \subseteq V \times \mathbb{S}.$$

Up to a multiplicative constant depending on $\delta(K)$ only, $\partial\psi$ equals the square of the probability that the odd part of ψ is death-free.

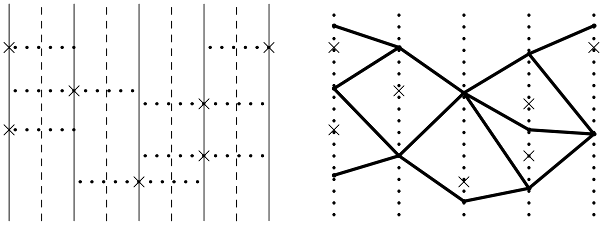

Fig. 2 *Left*: The partition $E(D)$. We have: $K_v = \mathbb{S}$ for $v \in V$, the *lines* $v \times K_v$ are drawn as *solid*, the *lines* $e \times K_e$ as *dashed*, and elements of D are marked as *crosses*. The endpoints of the $e \times J^e_{k,l}$ are the points where the *dotted lines* meet the *dashed lines*. *Right*: The graph $G(D)$. In this illustration, the *dotted lines* are the $v \times K_v$, and the *solid lines* are the edges of $G(D)$

3.2 Random-Parity Representation

The expectation $E(\partial\psi^A)$ is taken over the sets B, G, and over the randomization that takes place when $W \neq \varnothing$, that is, E denotes expectation with respect to the measure $d\mu_\lambda(B)d\mu_\gamma(G)dM_{B,G}$. The notation has been chosen to harmonize with that used in [7] in the discrete case: the expectation $E(\partial\psi^A)$ will play the role of the probability $P(\partial\underline{n} = A)$ of [7]. The main result of this section now follows.

Theorem 3.1 (Random-parity representation) *For any finite set $A \subseteq \overline{K}$ of sources*,

$$\langle\sigma_A\rangle = \frac{E(\partial\psi^A)}{E(\partial\psi^\varnothing)}. \tag{3.6}$$

We introduce a second random object in advance of proving this. Let $D \in \mathcal{F}$, the set of finite subsets of K, and recall that $K \setminus D$ is a disjoint union of intervals of the form $v \times J^k_v$. For each $e = uv \in E$, and each $1 \le k \le n(u)$ and $1 \le l \le n(v)$, let

$$J^e_{k,l} := J^u_k \cap J^v_l, \tag{3.7}$$

and

$$E(D) = \left\{e \times J^e_{k,l} : e \in E,\ 1 \le k \le n(u),\ 1 \le l \le n(v),\ J^e_{k,l} \neq \varnothing\right\}. \tag{3.8}$$

Up to a finite set of points, $E(D)$ forms a partition of the set F induced by the 'deaths' in D.

The pair

$$G(D) := (V(D), E(D)) \tag{3.9}$$

may be viewed as a graph, illustrated in Fig. 2. We will use the symbols $\bar{v}$ and $\bar{e}$ for typical elements of $V(D)$ and $E(D)$, respectively. There are natural weights on the edges and vertices of $G(D)$: for $\bar{e} = e \times J^e_{k,l} \in E(D)$ and $\bar{v} = v \times J^v_k \in V(D)$, let

$$J_{\bar{e}} := \int_{J^e_{k,l}} \lambda(e,t)\,dt, \qquad h_{\bar{v}} := \int_{J^v_k} \gamma(v,t)\,dt. \tag{3.10}$$

Thus the weight of a vertex or edge is its measure, calculated according to λ or γ, respectively. By (3.10),

$$\sum_{\bar{e}\in E(D)} J_{\bar{e}} + \sum_{\bar{v}\in V(D)} h_{\bar{v}} = \int_F \lambda(e)\,de + \int_K \gamma(x)\,dx, \quad D \in \mathcal{F}. \tag{3.11}$$

Proof With $\Lambda = (K, F)$ as in (3.2), we consider the partition function $Z' = Z'_K$ given in (3.4). For each $\bar{v} \in V(D)$, $\bar{e} \in E(D)$, the spins σ_v and σ_e are constant for $x \in \bar{v}$ and $e \in \bar{e}$, respectively. Denoting their common values by $\sigma_{\bar{v}}$ and $\sigma_{\bar{e}}$ respectively, the summation in (3.4) equals

$$\sum_{\sigma \in \Sigma(D)} \exp\left\{ \sum_{\bar{e} \in E(D)} \sigma_{\bar{e}} \int_{\bar{e}} \lambda(e)\, de + \sum_{\bar{v} \in V(D)} \sigma_{\bar{v}} \int_{\bar{v}} \gamma(x)\, dx \right\}$$
$$= \sum_{\sigma \in \Sigma(D)} \exp\left\{ \sum_{\bar{e} \in E(D)} J_{\bar{e}} \sigma_{\bar{e}} + \sum_{\bar{v} \in V(D)} h_{\bar{v}} \sigma_{\bar{v}} \right\}. \tag{3.12}$$

The right side of (3.12) is the partition function of the discrete Ising model on the graph $G(D)$, with pair couplings $J_{\bar{e}}$ and external fields $h_{\bar{v}}$. We shall apply the random-current expansion of [7] to this model.

For convenience of exposition, we introduce the extended graph

$$\begin{aligned} \widetilde{G}(D) &= (\widetilde{V}(D), \widetilde{E}(D)) \\ &:= \big(V(D) \cup \{\Gamma\}, E(D) \cup \{\bar{v}\Gamma : \bar{v} \in V(D)\}\big) \end{aligned} \tag{3.13}$$

where Γ is the ghost-site of (3.3). We call members of $E(D)$ *lattice-bonds*, and those of $\widetilde{E}(D) \setminus E(D)$ *ghost-bonds*. Let $\Psi(D)$ be the random multigraph with vertex set $\widetilde{V}(D)$ and with each edge of $\widetilde{E}(D)$ replaced by a random number of parallel edges, these numbers being independent and having the Poisson distribution, with parameter $J_{\bar{e}}$ for lattice-bonds $\bar{e}$, and parameter $h_{\bar{v}}$ for ghost-bonds $\bar{v}\Gamma$.

Let $\{\partial\Psi(D) = A\}$ denote the event that, for each $\bar{v} \in V(D)$, the total degree of $\bar{v}$ in $\Psi(D)$ *plus* the number of elements of A inside the closure of $\bar{v}$ (when regarded as an interval) is even. (There is μ_δ-probability 0 that $A \cap D \neq \varnothing$, and thus we may overlook this possibility.) Applying the discrete random-current expansion, and in particular [23, (9.24)], we obtain by (3.11) that

$$\sum_{\sigma \in \Sigma(D)} \exp\left\{ \sum_{\bar{e} \in E(D)} J_{\bar{e}} \sigma_{\bar{e}} + \sum_{\bar{v} \in V(D)} h_{\bar{v}} \sigma_{\bar{v}} \right\} = c 2^{|V(D)|} P_D(\partial\Psi(D) = \varnothing), \tag{3.14}$$

where P_D is the law of the edge-counts, and

$$c = \exp\left\{ \int_F \lambda(e)\, de + \int_K \gamma(x)\, dx \right\}. \tag{3.15}$$

By the same argument applied to the numerator in (2.4) (adapted to the measure on Λ, see the remark after (3.4)),

$$\langle \sigma_A \rangle = \frac{E(2^{|V(D)|} 1\{\partial\Psi(D) = A\})}{E(2^{|V(D)|} 1\{\partial\Psi(D) = \varnothing\})}, \tag{3.16}$$

where the expectation is with respect to $\mu_\delta \times P_D$. The claim of the theorem will follow by an appropriate manipulation of (3.16).

Here is another way to sample $\Psi(D)$ which allows us to couple it with the random colouring ψ^A. Let $B \in \mathcal{B}$ and $D, G \in \mathcal{F}$. The number of points of G lying in the interval

$\bar{v} = v \times J_k^v$ has the Poisson distribution with parameter $h_{\bar{v}}$, and similarly the number of elements of B lying in $\bar{e} = e \times J_{k,l}^e \in E(D)$ has the Poisson distribution with parameter $J_{\bar{e}}$. Thus, for given D, the multigraph $\Psi(B, G, D)$, obtained by replacing an edge of $\widetilde{E}(D)$ by parallel edges equal in number to the corresponding number of points from B or G, respectively, has the same law as $\Psi(D)$. Using the *same* sets B, G we may form the random colouring ψ^A.

The numerator of (3.16) satisfies

$$\begin{aligned}
&E(2^{|V(D)|}1\{\partial\Psi(D) = A\}) \\
&= \iint d\mu_\lambda(B)\, d\mu_\gamma(G) \int d\mu_\delta(D)\, 2^{|V(D)|} 1\{\partial\Psi(B, G, D) = A\} \\
&= \mu_\delta(2^{|V(D)|}) \iint d\mu_\lambda(B)\, d\mu_\gamma(G)\, \widetilde{\mu}(\partial\Psi(B, G, D) = A), \qquad (3.17)
\end{aligned}$$

where $\widetilde{\mu}$ is the probability measure on $\mathcal{F}$ satisfying

$$\frac{d\widetilde{\mu}}{d\mu_\delta}(D) \propto 2^{|V(D)|}. \qquad (3.18)$$

Therefore, by (3.16),

$$\langle\sigma_A\rangle = \frac{\widetilde{P}(\partial\Psi(B, G, D) = A)}{\widetilde{P}(\partial\Psi(B, G, D) = \varnothing)}, \qquad (3.19)$$

where $\widetilde{P}$ denotes the probability under $\mu_\lambda \times \mu_\gamma \times \widetilde{\mu}$. We claim that

$$\widetilde{\mu}(\partial\Psi(B, G, D) = A) = s M_{B,G}(\partial\psi^A(B, G)), \qquad (3.20)$$

for all B, G, where s is a constant, and the expectation $M_{B,G}$ is over the uniform measure on the set of valid colourings. Claim (3.6) follows from this, and the remainder of the proof is to show (3.20). The constants s, s_j are permitted in the following to depend only on Λ and δ.

Here is a special case. For $B \in \mathcal{B}$, $G \in \mathcal{F}$,

$$\widetilde{\mu}(\partial\Psi(B, G, D) = A) = 0 \qquad (3.21)$$

if and only if some interval $\overline{I_k^v}$ contains an odd number of switching points, if and only if $\psi^A(B, G) = \#$ and $\partial\psi^A(B, G) = 0$. Thus (3.20) holds in this case.

Another special case arises when $K_v = [0, \beta)$ for all $v \in V$, that is, the 'free boundary' case. Assume that each $\overline{K_v}$ contains an even number of switching points. As remarked earlier, there is a unique valid colouring $\psi^A = \psi^A(B, G)$. Moreover, $|V(D)| = |D| + |V|$, whence from standard properties of Poisson processes, $\widetilde{\mu} = \mu_{2\delta}$. It may be seen after some thought (possibly with the aid of a diagram) that, for given B, G, the events $\{\partial\Psi(B, G, D) = A\}$ and $\{D \cap \mathrm{odd}(\psi^A) = \varnothing\}$ differ by an event of $\mu_{2\delta}$-probability 0. Therefore,

$$\begin{aligned}
\widetilde{\mu}(\partial\Psi(B, G, D) = A) &= \mu_{2\delta}(D \cap \mathrm{odd}(\psi^A) = \varnothing) \\
&= \exp\{-2\delta(\mathrm{odd}(\psi^A))\} \\
&= s_1 \exp\{2\delta(\mathrm{ev}(\psi^A))\} = s_1 \partial\psi^A, \qquad (3.22)
\end{aligned}$$

with $s_1 = e^{-2\delta(K)}$. In this special case, (3.20) holds.

For the general case, we first note some properties of $\widetilde{\mu}$. By the above, we may assume that B, G are such that $\widetilde{\mu}(\partial\Psi(B,G,D)=A)>0$, which is to say that each $\overline{I_k^v}$ contains an even number of switching points. Let $W=\{v\in V: K_v=\mathbb{S}\}$ and, for $v\in V$, let $D_v=D\cap(v\times K_v)$ and $d(v)=|D_v|$. By (3.18),

$$\begin{aligned}\frac{d\widetilde{\mu}}{d\mu_\delta}(D)\propto 2^{|V(D)|}&=\prod_{w\in W}2^{1\vee d(w)}\prod_{v\in V\setminus W}2^{m(v)+d(v)}\\&\propto 2^{|D|}\prod_{w\in W}2^{1\{d(w)=0\}},\end{aligned}$$

where $a\vee b=\max\{a,b\}$, and we recall the number $m(v)$ of intervals I_k^v that constitute K_v. Therefore,

$$\frac{d\widetilde{\mu}}{d\mu_{2\delta}}(D)\propto\prod_{w\in W}2^{1\{d(w)=0\}}. \tag{3.23}$$

Three facts follow.

(a) The sets D_v, $v\in V$ are independent under $\widetilde{\mu}$.
(b) For $v\in V\setminus W$, the law of D_v under $\widetilde{\mu}$ is $\mu_{2\delta}$.
(c) For $w\in W$, the law μ_w of D_w is that of $\mu_{2\delta}$ skewed by the Radon–Nikodym factor $2^{1\{d(w)=0\}}$, which is to say that

$$\begin{aligned}\mu_w(D_w\in H)=&\frac{1}{\alpha_w}\big[2\mu_{2\delta}(D_w\in H,\, d(w)=0)\\&+\mu_{2\delta}(D_w\in H,\, d(w)\geq 1)\big],\end{aligned} \tag{3.24}$$

for appropriate sets H, where

$$\alpha_w=\mu_{2\delta}(d(w)=0)+1.$$

Recall the set $S=A\cup G\cup V(B)$ of switching points. By (a) above,

$$\begin{aligned}\widetilde{\mu}(\partial\Psi(B,G,D)=A)&=\widetilde{\mu}(\forall v,k: |S\cap\overline{J_k^v}|\text{ is even})\\&=\prod_{v\in V}\widetilde{\mu}(\forall k: |S\cap\overline{J_k^v}|\text{ is even}).\end{aligned} \tag{3.25}$$

We claim that

$$\widetilde{\mu}(\forall k: |S\cap\overline{J_k^v}|\text{ is even})=s_2(v)M_{B,G}\Big(\exp\big\{2\delta\big(\mathrm{ev}(\psi^A)\cap(v\times K_v)\big)\big\}\Big), \tag{3.26}$$

where $M_{B,G}$ is as before. Recall that $M_{B,G}$ is a product measure. Once (3.26) is proved, (3.20) follows by (3.5) and (3.25).

For $v\in V\setminus W$, the restriction of ψ^A to $v\times K_v$ is determined given B and G, whence by (b) above, and the remark prior to (3.22),

$$\begin{aligned}\widetilde{\mu}(\forall k: |S\cap\overline{J_k^v}|\text{ is even})&=\mu_{2\delta}(\forall k: |S\cap\overline{J_k^v}|\text{ is even})\\&=\exp\big\{-2\delta\big(\mathrm{odd}(\psi^A)\cap(v\times K_v)\big)\big\}.\end{aligned} \tag{3.27}$$

Equation (3.26) follows with $s_2(v)=\exp\{-2\delta(v\times K_v)\}$.

For $w \in W$, by (3.24),

$$
\begin{aligned}
&\widetilde{\mu}(\forall k : |S \cap \overline{J_k^w}| \text{ is even}) \\
&\quad = \frac{1}{\alpha_w}\big[2\mu_{2\delta}(D_w = \varnothing) + \mu_{2\delta}(D_w \neq \varnothing,\ \forall k : |S \cap \overline{J_k^w}| \text{ is even})\big] \\
&\quad = \frac{1}{\alpha_w}\big[\mu_{2\delta}(D_w = \varnothing) + \mu_{2\delta}(\forall k : |S \cap \overline{J_k^w}| \text{ is even})\big].
\end{aligned}
$$

Let $\psi = \psi^A(B, G)$ be a valid colouring with $\psi(w, 0) = \text{even}$. (If $(w, 0) \in A$, we take $\psi(w, 0+) = \text{even}$.) The colouring $\overline{\psi}$, obtained from ψ by flipping all colours on $w \times K_w$, is valid also. Taking into account the periodic boundary condition,

$$
\begin{aligned}
&\mu_{2\delta}(\forall k : |S \cap \overline{J_k^w}| \text{ is even}) \\
&\quad = \mu_{2\delta}\big(\{D_w \cap \text{odd}(\psi) = \varnothing\} \cup \{D_w \cap \text{ev}(\psi) = \varnothing\}\big) \\
&\quad = \mu_{2\delta}(D_w \cap \text{odd}(\psi) = \varnothing) + \mu_{2\delta}(D_w \cap \text{ev}(\psi) = \varnothing) - \mu_{2\delta}(D_w = \varnothing),
\end{aligned}
$$

whence

$$
\begin{aligned}
&\alpha_w \widetilde{\mu}(\forall k : |S \cap \overline{J_k^w}| \text{ is even}) \\
&\quad = \mu_{2\delta}(D_w \cap \text{odd}(\psi) = \varnothing) + \mu_{2\delta}(D_w \cap \text{ev}(\psi) = \varnothing) \\
&\quad = 2M_{B,G}\Big(\exp\big\{-2\delta\big(\text{odd}(\psi^A) \cap (w \times K_w)\big)\big\}\Big),
\end{aligned}
$$

since $\text{odd}(\psi^A) = \text{odd}(\psi)$ with $M_{B,G}$-probability $\frac{1}{2}$, and equals $\text{ev}(\psi)$ otherwise. This proves (3.26) with $s_2(w) = 2\exp\{-2\delta(w \times K_w)\}/\alpha_w$. □

By keeping track of the constants in the above proof, we arrive at the following statement, which will be useful later.

Lemma 3.2 *The partition function $Z' = Z'_K$ of (3.4) satisfies*

$$
Z' = 2^N e^{\lambda(F)+\gamma(K)-\delta(K)} E(\partial\psi^{\varnothing}),
$$

where $N = \sum_{v \in V} m(v)$ is the total number of intervals comprising K.

3.3 The Backbone

The concept of the backbone is key to the analysis of [7], and its definition there has a certain complexity. The corresponding definition is easier in the current setting, because of the fact that bridges, deaths, and sources have (almost surely) no common point.

We construct a total order on K by: first ordering the vertices of L, and then using the natural order on $[0, \beta)$. Let $A \subseteq \overline{K}$ be a finite set of sources, and let $B \in \mathcal{B}$, $G \in \mathcal{F}$. Let ψ be a valid colouring. We will define a sequence of directed odd paths called the *backbone* and denoted $\xi = \xi(\psi)$. Suppose $A = (a_1, a_2, \ldots, a_n)$ in the above ordering. Starting at a_1, follow the odd interval (in ψ) until you reach an element of $S = A \cup G \cup V(B)$. If the first such point

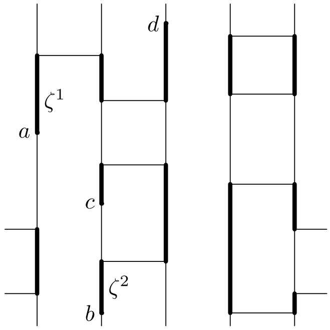

Fig. 3 A valid colouring configuration ψ with sources $A=\{a,b,c,d\}$, and its backbone $\xi=\zeta^1\circ\zeta^2$. Note that, in this illustration, bridges protruding from the sides 'wrap around', and that there are no ghost-bonds

thus encountered is the endpoint of a bridge, cross it, and continue along the odd interval; continue likewise until we first reach a point $t_1\in A\cup G$, at which point we stop. Note, by the validity of ψ, that $a_1\neq t_1$. The odd path thus traversed is denoted ζ^1; we take ζ^1 to be closed (when viewed as a subset of $\mathbb{Z}^d\times\mathbb{R}$). Repeat the same procedure with A replaced by $A\setminus\{a_1,t_1\}$, and iterate until no sources remain. The resulting (unordered) set of paths $\xi=(\zeta^1,\dots,\zeta^k)$ is called the *backbone* of ψ. The backbone will also be denoted at times as $\xi=\zeta^1\circ\cdots\circ\zeta^k$. We define $\xi(\#)=\varnothing$. Note that, apart from the backbone, the remaining odd segments of ψ form disjoint self-avoiding cycles (or 'eddies'). Unlike the discrete setting of [7], there is a (a.s.) unique way of specifying the backbone from knowledge of A, B, G and the valid colouring ψ. See Fig. 3.

The backbone contains all the sources A as endpoints, and the configuration outside ξ may be any sourceless configuration. Moreover, since ξ is entirely odd, it does not contribute to the weight $\partial\psi$ in (3.5). It follows, using properties of Poisson processes, that the conditional expectation $E(\partial\psi^A\mid\xi)$ equals the expected weight of any sourceless colouring of $K\setminus\xi$, which is to say that, with $\xi:=\xi(\psi^A)$,

$$E(\partial\psi^A\mid\xi)=E_{K\setminus\xi}(\partial\psi^\varnothing)=:Z_{K\setminus\xi}\tag{3.28}$$

cf. (3.4) and (3.6), and recall Remark 2.1. We abbreviate Z_K to Z, and recall from Lemma 3.2 that the Z_R differ from the partition functions Z'_R by certain multiplicative constants.

Let Ξ be the set of all possible backbones as A, B, and G vary, regarded as sequences of directed paths in K; these paths may, if required, be ordered by their starting points. For a source-set $A\subseteq\overline{K}$ and a backbone $\nu\in\Xi$, we write $A\sim\nu$ if there exists $B\in\mathcal{B}$ and $G\in\mathcal{F}$ such that $M_{B,G}(\xi(\psi^A)=\nu)>0$. We define the *weight* $w^A(\nu)$ by

$$w^A(\nu)=w^A_K(\nu):=\begin{cases}\frac{Z_{K\setminus\nu}}{Z}, & \text{if } A\sim\nu,\\ 0, & \text{otherwise.}\end{cases}\tag{3.29}$$

By (3.28) and Theorem 3.1, with $\xi=\xi(\psi^A)$,

$$E(w^A(\xi))=\frac{E(E(\partial\psi^A\mid\xi))}{Z}=\frac{E(\partial\psi^A)}{E(\partial\psi^\varnothing)}=\langle\sigma_A\rangle.\tag{3.30}$$

For $\nu^1,\nu^2\in\Xi$ with $\nu^1\cap\nu^2=\varnothing$ (that is, no point lies in paths of both ν^1 and ν^2), we write $\nu^1\circ\nu^2$ for the element of Ξ comprising the union of ν^1 and ν^2.

Let $\nu = \zeta^1 \circ \cdots \circ \zeta^k \in \Xi$ where $k \geq 1$. If ζ^i has starting point a_i and endpoint b_i, we write $\zeta^i : a_i \to b_i$, and also $\nu : a_1 \to b_1, \ldots, a_k \to b_k$. If $b_i \in G$, we write $\zeta^i : a_i \to \Gamma$. There is a natural way to 'cut' ν at points x lying on ζ^i, say, where $x \neq a_i, b_i$: let $\bar{\nu}^1 = \bar{\nu}^1(\nu, x) = \zeta^1 \circ \cdots \circ \zeta^{i-1} \circ \zeta^i_{\leq x}$ and $\bar{\nu}^2 = \bar{\nu}^2(\nu, x) = \zeta^i_{\geq x} \circ \zeta^{i+1} \circ \cdots \circ \zeta^k$, where $\zeta^i_{\leq x}$ (respectively, $\zeta^i_{\geq x}$) is the closed sub-path of ζ^i from a_i to x (respectively, x to b_i). We express this decomposition as $\nu = \bar{\nu}^1 \circ \bar{\nu}^2$ where, this time, each $\bar{\nu}^i$ may comprise a number of disjoint paths. The notation $\bar{\nu}$ will be used only in a situation where there has been a cut.

We note two special cases. If $A = \{a\}$, then necessarily $\xi(\psi^A) : a \to \Gamma$, so

$$\langle \sigma_a \rangle = E\big(w^a(\xi) \cdot 1\{\xi : a \to \Gamma\}\big). \tag{3.31}$$

If $A = \{a, b\}$ where $a < b$ in the ordering of K, then

$$\langle \sigma_a \sigma_b \rangle = E\big(w^{ab}(\xi) \cdot 1\{\xi : a \to b\}\big) + E\big(w^{ab}(\xi) \cdot 1\{\xi : a \to \Gamma,\ b \to \Gamma\}\big). \tag{3.32}$$

The last term equals 0 when $\gamma \equiv 0$.

Finally, here is a lemma for computing the weight of ν in terms of its constituent parts. The claim of the lemma is, as usual, valid only 'almost surely'.

Lemma 3.3
(a) *Let $\nu^1, \nu^2 \in \Xi$ be disjoint, and $\nu = \nu^1 \circ \nu^2$, $A \sim \nu$. Writing $A^i = A \cap \nu^i$, we have that*

$$w^A(\nu) = w^{A^1}(\nu^1) w^{A^2}_{K \setminus \nu^1}(\nu^2). \tag{3.33}$$

(b) *Let $\nu = \bar{\nu}^1 \circ \bar{\nu}^2$ be a cut of the backbone ν at the point x, and $A \sim \nu$. Then*

$$w^A(\nu) = w^{B^1}(\bar{\nu}^1) w^{B^2}_{K \setminus \bar{\nu}^1}(\bar{\nu}^2), \tag{3.34}$$

where $B^i = A^i \cup \{x\}$.

Proof By (3.29), the first claim is equivalent to

$$\frac{Z_{K \setminus \nu}}{Z} 1\{A \sim \nu\} = \frac{Z_{K \setminus \nu^1}}{Z} 1\{A^1 \sim \nu^1\} \frac{Z_{K \setminus (\nu^1 \cup \nu^2)}}{Z_{K \setminus \nu^1}} 1\{A^2 \sim \nu^2\}. \tag{3.35}$$

The right side vanishes if and only if the left side vanishes. When both sides are non-zero, their equality follows from the fact that $Z_{K \setminus \nu} = Z_{K \setminus (\nu^1 \cup \nu^2)}$. The second claim follows similarly, on adding x to the set of sources. □

4 The Switching Lemma

We state and prove next the principal tool in the random-parity representation, namely the so-called 'switching lemma'. In brief, this allows us to take two independent colourings, with different sources, and to 'switch' the sources from one to the other in a measure-preserving way. In so doing, the backbone will generally change. In order to preserve the measure, the *connectivities* inherent in the backbone must be retained. We begin by defining two notions of connectivity in colourings. We work throughout this section in the general set-up of Sect. 3.1.

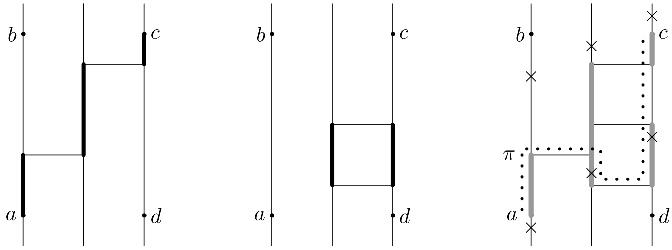

Fig. 4 Connectivity in pairs of colourings. *Left*: ψ_1^{ac}. *Middle*: $\psi_2^{\varnothing}$. *Right*: the triple $\psi_1^{ac}, \psi_2^{\varnothing}, \Delta$. *Crosses* are elements of Δ and *grey lines* are where either ψ_1^{ac} or $\psi_2^{\varnothing}$ is odd. In $(\psi_1^{ac}, \psi_2^{\varnothing}, \Delta)$ the following connectivities hold: $a \not\leftrightarrow b$, $a \leftrightarrow c$, $a \leftrightarrow d$, $b \not\leftrightarrow c$, $b \not\leftrightarrow d$, $c \leftrightarrow d$. The *dotted line* marks π, one of the open paths from a to c

4.1 Connectivity and Switching

Let $B \in \mathcal{B}$, $G \in \mathcal{F}$, let $A \subseteq \overline{K}$ be a finite set of sources, and write $\psi^A = \psi^A(B, G)$ for the colouring given in the last section. In what follows we think of the ghost-bonds as bridges to the ghost-site Γ.

Let $x, y \in K^\Gamma := K \cup \{\Gamma\}$. A *path* from x to y in the configuration (B, G) is a self-avoiding path with endpoints x, y, traversing intervals of K^Γ, and possibly bridges in B and/or ghost-bonds joining G to Γ. Similarly, a *cycle* is a self-avoiding cycle in the above graph. A *route* is a path or a cycle. A route containing no ghost-bonds is called a *lattice-route*. A route is called *odd* (in the colouring ψ^A) if ψ^A, when restricted to the route, takes only the value 'odd'. The failed colouring $\psi^A = \#$ is deemed to contain no odd routes.

Let $B_1, B_2 \in \mathcal{B}$, $G_1, G_2 \in \mathcal{F}$, and let $\psi_1^A = \psi_1^A(B_1, G_1)$ and $\psi_2^B = \psi_2^B(B_2, G_2)$ be the associated colourings. Let Δ be an auxiliary Poisson process on K, with intensity function $4\delta(\cdot)$, that is independent of all other random variables so far. We call points of Δ *cuts*. A route of $(B_1 \cup B_2, G_1 \cup G_2)$ is said to be *open* in the triple $(\psi_1^A, \psi_2^B, \Delta)$ if it includes no sub-interval of $\mathrm{ev}(\psi_1^A) \cap \mathrm{ev}(\psi_2^B)$ containing one or more elements of Δ. In other words, the cuts break paths, but only when they belong to intervals labelled 'even' in *both* colourings. See Fig. 4. In particular, if there is an odd path π from x to y in ψ_1^A, then π constitutes an open path in $(\psi_1^A, \psi_2^B, \Delta)$ irrespective of ψ_2^B and Δ. We let

$$\{x \leftrightarrow y \text{ in } \psi_1^A, \psi_2^B, \Delta\} \tag{4.1}$$

be the event that there exists an open path from x to y in $(\psi_1^A, \psi_2^B, \Delta)$. We may abbreviate this to $\{x \leftrightarrow y\}$ when there is no ambiguity.

There is an analogy between open paths in the above construction and the notion of connectivity in the random-current representation of the discrete Ising model. Points labelled 'odd' or 'even' above may be considered as collections of infinitesimal parallel edges, being odd or even in number, respectively. If a point is 'even', the corresponding number of edges may be $2, 4, 6, \dots$ *or* it may be 0; in the 'union' of ψ_1^A and ψ_2^B, connectivity is broken at a point if and only if both the corresponding numbers equal 0. It turns out that the correct law for the set of such points is that of Δ.

Here is some notation. For any finite sequence $(a, b, c, \dots)$ of elements in K, the string $abc\dots$ will denote the subset of elements that appear an odd number of times in the sequence. If $A \subseteq \overline{K}$ is a finite source-set with odd cardinality, then for any pair (B, G) for which there exists a valid colouring $\psi^A(B, G)$, the number of ghost-bonds must be odd.

Thinking of these as bridges to Γ, Γ may thus be viewed as an element of A, and we make the following remark.

Remark 4.1 For a source-set $A \subseteq \overline{K}$ with $|A|$ odd, we shall use the expressions ψ^A and $\psi^{A\cup\{\Gamma\}}$ interchangeably.

We call a function F, acting on $(\psi_1^A, \psi_2^B, \Delta)$, a *connectivity function* if it depends only on the connectivity properties using open paths of $(\psi_1^A, \psi_2^B, \Delta)$, that is, the value of F depends only on the set $\{(x, y) \in (K^\Gamma)^2 : x \leftrightarrow y\}$. In the following, E denotes expectation with respect to $d\mu_\lambda\, d\mu_\gamma\, dM_{B,G}\, dP$ where P is the law of Δ.

Theorem 4.2 (Switching lemma) *Let F be a connectivity function and $A, B \subseteq \overline{K}$ finite source-sets. For $x, y \in \overline{K} \cup \{\Gamma\}$ such that $A \triangle xy$ and $B \triangle xy$ are source-sets,*

$$\begin{aligned} &E\big(\partial\psi_1^A \partial\psi_2^B \cdot F(\psi_1^A, \psi_2^B, \Delta) \cdot 1\{x \leftrightarrow y \textit{ in } \psi_1^A, \psi_2^B, \Delta\}\big) \\ &\quad = E\Big(\partial\psi_1^{A\triangle xy} \partial\psi_2^{B\triangle xy} \cdot F(\psi_1^{A\triangle xy}, \psi_2^{B\triangle xy}, \Delta) \\ &\qquad \times 1\{x \leftrightarrow y \textit{ in } \psi_1^{A\triangle xy}, \psi_2^{B\triangle xy}, \Delta\}\Big). \end{aligned} \tag{4.2}$$

In particular,

$$E(\partial\psi_1^{xy} \partial\psi_2^B) = E\big(\partial\psi_1^{\varnothing} \partial\psi_2^{B\triangle xy} \cdot 1\{x \leftrightarrow y \textit{ in } \psi_1^{\varnothing}, \psi_2^{B\triangle xy}, \Delta\}\big). \tag{4.3}$$

Proof Equation (4.3) follows from (4.2) with $A = \{x, y\}$ and $F \equiv 1$, and so it suffices to prove (4.2). This is trivial if $x = y$, and we assume henceforth that $x \neq y$. Recall that $W = \{v \in V : K_v = \mathbb{S}\}$ and $|W| = r$.

We prove (4.2) first for the special case when $F \equiv 1$, that is,

$$\begin{aligned} &E\big(\partial\psi_1^A \partial\psi_2^B \cdot 1\{x \leftrightarrow y \text{ in } \psi_1^A, \psi_2^B, \Delta\}\big) \\ &\quad = E\big(\partial\psi_1^{A\triangle xy} \partial\psi_2^{B\triangle xy} \cdot 1\{x \leftrightarrow y \text{ in } \psi_1^{A\triangle xy}, \psi_2^{B\triangle xy}, \Delta\}\big), \end{aligned} \tag{4.4}$$

and this will follow by conditioning on the pair $Q = (B_1 \cup B_2, G_1 \cup G_2)$.

Let $Q \in \mathcal{B} \times \mathcal{F}$ be given. Conditional on Q, the law of (ψ_1^A, ψ_2^B) is given as follows. First, we allocate each bridge and each ghost-bond to either ψ_1^A or ψ_2^B with equal probability (independently of one another). If $W \neq \varnothing$, then we must also allocate (uniform) random colours to the points $(w, 0)$, $w \in W$, for each of ψ_1^A, ψ_2^B. If $(w, 0)$ is itself a source, we work with $(w, 0+)$. (Recall that the pair (B', G') may be reconstructed from knowledge of a valid colouring $\psi^{A'}(B', G')$.) There are $2^{|Q|+2r}$ possible outcomes of the above choices, and each is equally likely.

The process Δ is independent of all random variables used above. Therefore, the conditional expectation, given Q, of the random variable on the left side of (4.4) equals

$$\frac{1}{2^{|Q|+2r}} \sum_{Q^{A,B}} \partial Q_1 \partial Q_2\, P(x \leftrightarrow y \text{ in } Q_1, Q_2, \Delta), \tag{4.5}$$

where the sum is over the set $Q^{A,B} = Q^{A,B}(Q)$ of all possible pairs (Q_1, Q_2) of values of (ψ_1^A, ψ_2^B). The measure P is that of Δ.

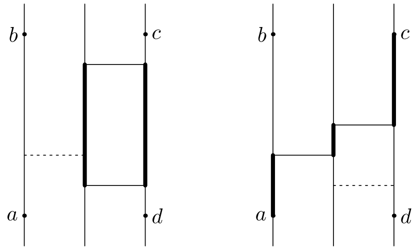

Fig. 5 Switched configurations. Taking Q_1^{ac}, $Q_2^{\varnothing}$ and π to be ψ_1^{ac}, $\psi_2^{\varnothing}$ and π of Fig. 4, respectively, this figure illustrates the 'switched' configurations $R_1^{\varnothing}$ and R_2^{ac} (*left* and *right*, respectively)

We shall define an invertible (and therefore measure-preserving) map from $\mathcal{Q}^{A,B}$ to $\mathcal{Q}^{A\triangle xy,B\triangle xy}$. Let π be a path of Q with endpoints x and y (if such a path π exists), and let $f_\pi : \mathcal{Q}^{A,B} \to \mathcal{Q}^{A\triangle xy,B\triangle xy}$ be given as follows. Let $(Q_1, Q_2) \in \mathcal{Q}^{A,B}$, say $Q_1 = Q_1^A(B_1, G_1)$ and $Q_2 = Q_2^B(B_2, G_2)$ where $Q = (B_1 \cup B_2, G_1 \cup G_2)$. For $i = 1, 2$, let B_i' (respectively, G_i') be the set of bridges (respectively, ghost-bonds) in Q lying in exactly one of B_i, π (respectively, G_i, π). Otherwise expressed, (B_i', G_i') is obtained from (B_i, G_i) by adding the bridges/ghost-bonds of π 'modulo 2'. Note that $(B_1' \cup B_2', G_1' \cup G_2') = Q$.

If $W = \varnothing$, we let $R_1 = R_1^{A\triangle xy}$ (respectively, $R_2^{B\triangle xy}$) be the unique valid colouring of (B_1', G_1') with sources $A \triangle xy$ (respectively, (B_2', G_2') with sources $B \triangle xy$), so $R_1 = \psi^{A\triangle xy}(B_1', G_1')$, and similarly for R_2. When $W \neq \varnothing$ and $i = 1, 2$, we choose the colours of the $(w, 0)$, $w \in W$, (or $(w, 0+)$ if $(w, 0)$ is a source) in R_i in such a way that $R_i \equiv Q_i$ on $K \setminus \pi$.

It is easily seen that the map $f_\pi : (Q_1, Q_2) \mapsto (R_1, R_2)$ is invertible, indeed its inverse is given by the same mechanism. See Fig. 5.

By (3.5),

$$\partial Q_1 \partial Q_2 = \exp\bigl\{2\delta(\mathrm{ev}(Q_1)) + 2\delta(\mathrm{ev}(Q_2))\bigr\}. \tag{4.6}$$

Now,

$$\begin{aligned} \delta(\mathrm{ev}(Q_i)) &= \delta(\mathrm{ev}(Q_i) \cap \pi) + \delta(\mathrm{ev}(Q_i) \setminus \pi) \\ &= \delta(\mathrm{ev}(Q_i) \cap \pi) + \delta(\mathrm{ev}(R_i) \setminus \pi), \end{aligned} \tag{4.7}$$

and

$$\begin{aligned} &\delta(\mathrm{ev}(Q_1) \cap \pi) + \delta(\mathrm{ev}(Q_2) \cap \pi) - 2\delta\bigl(\mathrm{ev}(Q_1) \cap \mathrm{ev}(Q_2) \cap \pi\bigr) \\ &\quad = \delta\bigl(\mathrm{ev}(Q_1) \cap \mathrm{odd}(Q_2) \cap \pi\bigr) + \delta\bigl(\mathrm{odd}(Q_1) \cap \mathrm{ev}(Q_2) \cap \pi\bigr) \\ &\quad = \delta\bigl(\mathrm{odd}(R_1) \cap \mathrm{ev}(R_2) \cap \pi\bigr) + \delta\bigl(\mathrm{ev}(R_1) \cap \mathrm{odd}(R_2) \cap \pi\bigr) \\ &\quad = \delta(\mathrm{ev}(R_1) \cap \pi) + \delta(\mathrm{ev}(R_2) \cap \pi) - 2\delta\bigl(\mathrm{ev}(R_1) \cap \mathrm{ev}(R_2) \cap \pi\bigr), \end{aligned}$$

whence, by (4.6)–(4.7),

$$\begin{aligned} \partial Q_1 \partial Q_2 &= \partial R_1 \partial R_2 \exp\bigl\{-4\delta\bigl(\mathrm{ev}(R_1) \cap \mathrm{ev}(R_2) \cap \pi\bigr)\bigr\} \\ &\quad \times \exp\bigl\{4\delta\bigl(\mathrm{ev}(Q_1) \cap \mathrm{ev}(Q_2) \cap \pi\bigr)\bigr\}. \end{aligned} \tag{4.8}$$

The next step is to choose a suitable path π. Consider the final term in (4.5), namely

$$P(x \leftrightarrow y \text{ in } Q_1, Q_2, \Delta). \tag{4.9}$$

There are finitely many paths in Q from x to y, let these paths be $\pi_1, \pi_2, \dots, \pi_n$. Let $\mathcal{O}_k = \mathcal{O}_k(Q_1, Q_2, \Delta)$ be the event that π_k is the earliest such path that is open in (Q_1, Q_2, Δ). Then

$$
\begin{aligned}
P&(x \leftrightarrow y \text{ in } Q_1, Q_2, \Delta)\\
&= \sum_{k=1}^{n} P(\mathcal{O}_k)\\
&= \sum_{k=1}^{n} P\big(\Delta \cap [\mathrm{ev}(Q_1) \cap \mathrm{ev}(Q_2) \cap \pi_k] = \varnothing\big) P(\widetilde{\mathcal{O}}_k)\\
&= \sum_{k=1}^{n} \exp\big\{-4\delta\big(\mathrm{ev}(Q_1) \cap \mathrm{ev}(Q_2) \cap \pi_k\big)\big\} P(\widetilde{\mathcal{O}}_k),
\end{aligned}
\tag{4.10}
$$

where $\widetilde{\mathcal{O}}_k = \widetilde{\mathcal{O}}_k(Q_1, Q_2, \Delta)$ is the event (that is, subset of $\mathcal{F}$) that each of $\pi_1, \dots, \pi_{k-1}$ is rendered non-open in (Q_1, Q_2, Δ) through the presence of elements of Δ lying in $K \setminus \pi_k$. In the second line of (4.10), we have used the independence of $\Delta \cap \pi_k$ and $\Delta \cap (K \setminus \pi_k)$.

Let $(R_1^k, R_2^k) = f_{\pi_k}(Q_1, Q_2)$. Since $R_i^k \equiv Q_i$ on $K \setminus \pi_k$, we have that $\widetilde{\mathcal{O}}_k(Q_1, Q_2, \Delta) = \widetilde{\mathcal{O}}_k(R_1^k, R_2^k, \Delta)$. By (4.8) and (4.10), the summand in (4.5) equals

$$
\begin{aligned}
&\sum_{k=1}^{n} \partial Q_1 \partial Q_2 \exp\big\{-4\delta\big(\mathrm{ev}(Q_1) \cap \mathrm{ev}(Q_2) \cap \pi_k\big)\big\} P(\widetilde{\mathcal{O}}_k)\\
&\quad= \sum_{k=1}^{n} \partial R_1^k \partial R_2^k \exp\big\{-4\delta\big(\mathrm{ev}(R_1^k) \cap \mathrm{ev}(R_2^k) \cap \pi_k\big)\big\} P(\widetilde{\mathcal{O}}_k)\\
&\quad= \sum_{k=1}^{n} \partial R_1^k \partial R_2^k \, P(\mathcal{O}_k(R_1^k, R_2^k, \Delta)).
\end{aligned}
$$

Summing the above over $\mathcal{Q}^{A,B}$, and remembering that each f_{π_k} is a bijection between $\mathcal{Q}^{A,B}$ and $\mathcal{Q}^{A\triangle xy, B\triangle xy}$, (4.5) becomes

$$
\begin{aligned}
&\frac{1}{2^{|Q|+2r}} \sum_{k=1}^{n} \sum_{(R_1,R_2)\in \mathcal{Q}^{A\triangle xy, B\triangle xy}} \partial R_1 \partial R_2 \, P(\mathcal{O}_k(R_1, R_2, \Delta))\\
&\quad= \frac{1}{2^{|Q|+2r}} \sum_{\mathcal{Q}^{A\triangle xy, B\triangle xy}} \partial R_1 \partial R_2 \, P(x \leftrightarrow y \text{ in } R_1, R_2, \Delta).
\end{aligned}
$$

By the argument leading to (4.5), this equals the right side of (4.4), and the claim is proved when $F \equiv 1$.

Consider now the case of general connectivity functions F in (4.2). In (4.5), the factor $P(x \leftrightarrow y \text{ in } Q_1, Q_2, \Delta)$ is replaced by

$$
P\big(F(Q_1, Q_2, \Delta) \cdot 1\{x \leftrightarrow y \text{ in } Q_1, Q_2, \Delta\}\big),
$$

where P denotes expectation with respect to Δ. In the calculation (4.10), we use the fact that

$$
P(F \cdot 1_{\mathcal{O}_k}) = P(F \mid \mathcal{O}_k) P(\mathcal{O}_k)
$$

and we deal with the factor $P(\mathcal{O}_k)$ as before. The result follows on noting that, for each k,

$$P\big(F(Q_1, Q_2, \Delta)\,\big|\,\mathcal{O}_k(Q_1, Q_2, \Delta)\big) = P\big(F(R_1^k, R_2^k, \Delta)\,\big|\,\mathcal{O}_k(R_1^k, R_2^k, \Delta)\big).$$

This holds because: (i) the configurations (Q_1, Q_2, Δ) and (R_1^k, R_2^k, Δ) are identical off π_k, and (ii) in each, all points along π_k are connected. Thus the connectivities are identical in the two configurations. □

4.2 Applications of Switching

In this section are presented a number of inequalities and identities proved using the random-parity representation and the switching lemma. With some exceptions (most notably (4.40)) the proofs are adaptations of the proofs for the discrete Ising model that may be found in [7, 23].

For functions $f, g : K \to \mathbb{R}$, we write $f \le g$ if $f(x) \le g(x)$ for all $x \in K$.

Lemma 4.3 (GKS inequality) *Let $A, B \subseteq \overline{K}$ be finite sets of sources, not necessarily disjoint. Then*

$$\langle \sigma_A \rangle \ge 0, \tag{4.11}$$

and

$$\langle \sigma_A; \sigma_B \rangle := \langle \sigma_A \sigma_B \rangle - \langle \sigma_A \rangle \langle \sigma_B \rangle \ge 0. \tag{4.12}$$

Lemma 4.4 *Let $A \subseteq \overline{K}$ be a finite set of sources. Then $\langle \sigma_A \rangle$ is increasing in λ and γ and decreasing in δ. Moreover, if $R \subseteq K$ is measurable,*

$$\langle \sigma_A \rangle_{K \setminus R} \le \langle \sigma_A \rangle_K. \tag{4.13}$$

We interpret $\langle \sigma_A \rangle_{K \setminus R}$ as 0 when A is not a source-set for $K \setminus R$.

Lemmas 4.3 and 4.4 may be shown using conventional inequalities of spin-correlation-type. They may be proved more easily using the FKG-inequality for the associated random-cluster model (using, for example, the methods of [26]). We omit these proofs, full details of which may be found in [12].

For $R \subseteq K$ a finite union of intervals, let

$$\widetilde{R} := \{(uv, t) \in F : \text{either } (u, t) \in R \text{ or } (v, t) \in R \text{ or both}\}.$$

Recall that $W = W(K) = \{v \in V : K_v = \mathbb{S}\}$, and $N = N(K)$ is the total number of (maximal) intervals constituting K.

Lemma 4.5 *Let $R \subseteq K$ be a finite union of intervals, and let $\nu \in \Xi$ be such that $\nu \cap R = \varnothing$. If $A \subseteq \overline{K \setminus R}$ is a finite source-set for both K and $K \setminus R$, and $A \sim \nu$, then*

$$w^A(\nu) \le 2^{r(\nu) - r'(\nu)} w^A_{K \setminus R}(\nu), \tag{4.14}$$

where

$$\begin{aligned} r(\nu) &= r(\nu, K) := |\{w \in W : \nu \cap (w \times K_w) \ne \varnothing\}|, \\ r'(\nu) &= r(\nu, K \setminus R). \end{aligned}$$

Proof By (3.29) and Lemma 3.2,

$$
\begin{aligned}
w^A(\nu) &= \frac{Z_{K\setminus\nu}}{Z_K} \\
&= 2^{N(K)-N(K\setminus\nu)} e^{\lambda(\widetilde{\nu})+\gamma(\nu)-\delta(\nu)} \frac{Z'_{K\setminus\nu}}{Z'_K}.
\end{aligned}
\tag{4.15}
$$

We claim that

$$
\frac{Z'_{K\setminus\nu}}{Z'_K} \le \frac{Z'_{K\setminus(R\cup\nu)}}{Z'_{K\setminus R}},
\tag{4.16}
$$

and the proof of this follows.

Recall the formula (3.4) for Z'_K in terms of an integral over the Poisson process D. The set D is the union of independent Poisson processes D' and D'', restricted respectively to $K \setminus \nu$ and ν. We write P' (respectively, P'') for the probability measure (and, on occasion, expectation operator) governing D' (respectively, D''). Let $\Sigma(D')$ denote the set of spin configurations on $K \setminus \nu$ that are permitted by D'. By (3.4),

$$
Z'_K = P'\left(\sum_{\sigma'\in\Sigma(D')} Z'_\nu(\sigma') \exp\left\{ \int_{F\setminus\widetilde{\nu}} \lambda(e)\sigma'_e\, de + \int_{K\setminus\nu} \gamma(x)\sigma'_x\, dx \right\} \right),
\tag{4.17}
$$

where

$$
Z'_\nu(\sigma') = P''\left(\sum_{\sigma''\in\widetilde{\Sigma}(D'')} \exp\left\{ \int_{\widetilde{\nu}} \lambda(e)\sigma_e\, de + \int_\nu \gamma(x)\sigma_x\, dx \right\} \cdot 1_C(\sigma') \right)
$$

is the partition function on ν with boundary condition σ', and where σ, $\widetilde{\Sigma}(D'')$, and $C = C(D'')$ are given as follows.

The set D'' divides ν, in the usual way, into a collection $V_\nu(D'')$ of intervals. From the set of endpoints of such intervals, we distinguish the subset $\mathcal{E}$ that: (i) lie in K, and (ii) are endpoints of some interval of $K \setminus \nu$. For $x \in \mathcal{E}$, let $\sigma'_x = \lim_{y\to x} \sigma'_y$, where the limit is taken over $y \in K \setminus \nu$. Let $\widetilde{V}_\nu(D'')$ be the subset of $V_\nu(D'')$ containing those intervals with no endpoint in $\mathcal{E}$, and let $\widetilde{\Sigma}(D'') = \{-1,+1\}^{\widetilde{V}_\nu(D'')}$.

Let $\sigma' \in \Sigma(D')$, and let $\mathcal{I}$ be the set of maximal sub-intervals I of ν having both endpoints in $\mathcal{E}$, and such that $I \cap D'' = \varnothing$. Let $C = C(D'')$ be the set of $\sigma' \in \Sigma(D')$ such that, for all $I \in \mathcal{I}$, the endpoints of I have equal spins under σ'. Note that

$$
1_C(\sigma') = \prod_{I\in\mathcal{I}} \tfrac{1}{2}(\sigma'_{x(I)}\sigma'_{y(I)} + 1),
\tag{4.18}
$$

where $x(I)$, $y(I)$ denote the endpoints of I.

Let $\sigma'' \in \widetilde{\Sigma}(D'')$. The conjunction σ of σ' and σ'' is defined except on sub-intervals of ν lying in $V_\nu(D'') \setminus \widetilde{V}_\nu(D'')$. On any such sub-interval with exactly one endpoint x in $\mathcal{E}$, we set $\sigma \equiv \sigma'_x$. On the event C, an interval of ν with both endpoints $x(I)$, $y(I)$ in $\mathcal{E}$ receives the spin $\sigma \equiv \sigma'_{x(I)} = \sigma'_{y(I)}$. Thus, $\sigma \in \Sigma(D' \cup D'')$ is well defined for $\sigma' \in C$.

By (4.17),

$$\frac{Z'_K}{Z'_{K\setminus\nu}} = \langle Z'_\nu(\sigma')\rangle_{K\setminus\nu}.$$

Taking the expectation $\langle\cdot\rangle_{K\setminus\nu}$ inside the integral, the last expression becomes

$$P''\left(\sum_{\sigma''\in\widetilde{\Sigma}(D'')}\left\langle \exp\left\{\int_{\widetilde{\nu}}\lambda(e)\sigma_e\,de\right\}\exp\left\{\int_\nu \gamma(x)\sigma_x\,dx\right\}\cdot 1_C(\sigma')\right\rangle_{K\setminus\nu}\right).$$

The inner expectation may be expressed as a sum over $k,l\geq 0$ (with non-negative coefficients) of iterated integrals of the form

$$\frac{1}{k!}\frac{1}{l!}\iint\limits_{\widetilde{\nu}^k\times\nu^l}\lambda(\mathbf{e})\gamma(\mathbf{x})\langle\sigma_{e_1}\cdots\sigma_{e_k}\sigma_{x_1}\cdots\sigma_{x_l}\cdot 1_C(\sigma')\rangle_{K\setminus\nu}\,d\mathbf{e}\,d\mathbf{x}, \tag{4.19}$$

where we have written $\mathbf{e}=(e_1,\dots,e_k)$, and $\lambda(\mathbf{e})$ for $\lambda(e_1)\cdots\lambda(e_k)$ (and similarly for $\mathbf{x}$ and $\gamma(\mathbf{x})$). We may write

$$\langle\sigma_{e_1}\cdots\sigma_{e_k}\sigma_{x_1}\cdots\sigma_{x_l}\cdot 1_C\rangle_{K\setminus\nu} = \langle\sigma'_S\sigma''_T\cdot 1_C\rangle_{K\setminus\nu} = \sigma''_T\langle\sigma'_S\cdot 1_C\rangle_{K\setminus\nu},$$

for sets $S\subseteq\overline{K\setminus\nu}$, $T\subseteq\nu$ determined by $e_1,\dots,e_k,x_1,\dots,x_l$ and D'' only. We now bring the sum over σ'' inside the integral of (4.19). For $T\neq\varnothing$,

$$\sum_{\sigma''\in\widetilde{\Sigma}(D'')}\sigma''_T\langle\sigma'_S\cdot 1_C\rangle_{K\setminus\nu}=0,$$

so any non-zero term is of the form

$$\langle\sigma'_S\cdot 1_C\rangle_{K\setminus\nu}. \tag{4.20}$$

By (4.18), (4.20) may be expressed in the form

$$\sum_{i=1}^{s}2^{-a_i}\langle\sigma'_{S_i}\rangle_{K\setminus\nu} \tag{4.21}$$

for appropriate sets S_i and integers a_i. By Lemma 4.4,

$$\langle\sigma'_{S_i}\rangle_{K\setminus\nu}\geq\langle\sigma'_{S_i}\rangle_{K\setminus(R\cup\nu)}.$$

On working backwards, we obtain (4.16).

By (4.15)–(4.16),

$$w^A(\nu)\leq 2^U w^A_{K\setminus R}(\nu),$$

where

$$\begin{aligned}U&=\big[N(K)-N(K\setminus\nu)\big]-\big[N(K\setminus R)-N(K\setminus(R\cup\nu))\big]\\&=r(\nu)-r'(\nu)\end{aligned}$$

as required. □

For distinct $x, y, z \in K^\Gamma$, let

$$\begin{aligned}\langle \sigma_x; \sigma_y; \sigma_z\rangle :=& \langle \sigma_{xyz}\rangle - \langle \sigma_x\rangle\langle\sigma_{yz}\rangle \\ & - \langle\sigma_y\rangle\langle\sigma_{xz}\rangle - \langle\sigma_z\rangle\langle\sigma_{xy}\rangle + 2\langle\sigma_x\rangle\langle\sigma_y\rangle\langle\sigma_z\rangle.\end{aligned}$$

Lemma 4.6 (GHS inequality) *For distinct $x, y, z \in K^\Gamma$,*

$$\langle \sigma_x; \sigma_y; \sigma_z\rangle \le 0. \tag{4.22}$$

Moreover, $\langle\sigma_x\rangle$ is concave in γ in the sense that, for bounded, measurable functions $\gamma_1, \gamma_2 : K \to \mathbb{R}_+$ satisfying $\gamma_1 \le \gamma_2$, and $\theta \in [0, 1]$,

$$\theta\langle\sigma_x\rangle_{\gamma_1} + (1-\theta)\langle\sigma_x\rangle_{\gamma_2} \le \langle\sigma_x\rangle_{\theta\gamma_1+(1-\theta)\gamma_2}. \tag{4.23}$$

Proof The proof of this follows very closely the corresponding proof for the classical Ising model [21]. We include it here because it allows us to develop the technique of 'conditioning on clusters', which will be useful later.

We prove (4.22) via the following more general result. Let (B_i, G_i), $i = 1, 2, 3$, be independent sets of bridges/ghost-bonds, and write ψ_i, $i = 1, 2, 3$, for corresponding colourings (with sources to be specified through their superscripts). We claim that, for any four points $w, x, y, z \in K^\Gamma$,

$$\begin{aligned}&E\big(\partial\psi_1^\varnothing\partial\psi_2^\varnothing\partial\psi_3^{wxyz}\big) - E\big(\partial\psi_1^\varnothing\partial\psi_2^{wz}\partial\psi_3^{xy}\big)\\ &\quad\le E(\partial\psi_1^\varnothing\partial\psi_2^{wx}\partial\psi_3^{yz}) + E(\partial\psi_1^\varnothing\partial\psi_2^{wy}\partial\psi_3^{xz}) - 2E(\partial\psi_1^{wx}\partial\psi_2^{wy}\partial\psi_3^{wz}).\end{aligned} \tag{4.24}$$

Inequality (4.22) follows by Theorem 3.1 on letting $w = \Gamma$.

The left side of (4.24) is

$$\begin{aligned}&E(\partial\psi_1^\varnothing)\big[E(\partial\psi_2^\varnothing\partial\psi_3^{wxyz}) - E(\partial\psi_2^{wz}\partial\psi_3^{xy})\big]\\ &\quad = Z\, E\big(\partial\psi_2^\varnothing\partial\psi_3^{wxyz}\cdot 1\{w \nleftrightarrow z\}\big),\end{aligned}$$

by the Switching Lemma 4.2. When $\partial\psi_3^{wxyz}$ is non-zero, parity constraints imply that at least one of $\{w \leftrightarrow x\} \cap \{y \leftrightarrow z\}$ and $\{w \leftrightarrow y\} \cap \{x \leftrightarrow z\}$ occurs, but that, in the presence of the indicator function they cannot both occur. Therefore,

$$\begin{aligned}&E(\partial\psi_2^\varnothing\partial\psi_3^{wxyz}\cdot 1\{w \nleftrightarrow z\})\\ &\quad = E\big(\partial\psi_2^\varnothing\partial\psi_3^{wxyz}\cdot 1\{w \nleftrightarrow z\}\cdot 1\{w \leftrightarrow x\}\big)\\ &\qquad + E\big(\partial\psi_2^\varnothing\partial\psi_3^{wxyz}\cdot 1\{w \nleftrightarrow z\}\cdot 1\{w \leftrightarrow y\}\big).\end{aligned} \tag{4.25}$$

Consider the first term. By the switching lemma,

$$E\big(\partial\psi_2^\varnothing\partial\psi_3^{wxyz}\cdot 1\{w \nleftrightarrow z\}\cdot 1\{w \leftrightarrow x\}\big) = E\big(\partial\psi_2^{wx}\partial\psi_3^{yz}\cdot 1\{w \nleftrightarrow z\}\big). \tag{4.26}$$

We next 'condition on a cluster'. Let $C_z = C_z(\psi_2^{wx}, \psi_3^{yz}, \Delta)$ be the set of all points of K that are connected by open paths to z. Conditional on C_z, define new independent colourings $\mu_2^\varnothing$, μ_3^{yz} on the domain $M = C_z$. Similarly, let ν_2^{wx}, $\nu_3^\varnothing$ be independent colourings on the domain $N = K \setminus C_z$, that are also independent of the μ_i. It is not hard to see that, if $w \nleftrightarrow z$ in $(\psi_2^{wx}, \psi_3^{yz}, \Delta)$, then, conditional on C_z, the law of ψ_2^{wx} equals that of the superposition of $\mu_2^\varnothing$ and ν_2^{wx}; similarly the conditional law of ψ_3^{yz} is the same as that of the superposition of μ_3^{yz} and $\nu_3^\varnothing$. Therefore, almost surely on the event $\{w \nleftrightarrow z\}$,

$$\begin{aligned} E(\partial\psi_2^{wx}\partial\psi_3^{yz} \mid C_z) &= E'(\partial\mu_2^\varnothing)E'(\partial\nu_2^{wx})E'(\partial\mu_3^{yz})E'(\partial\nu_3^\varnothing) \\ &= \langle\sigma_{wx}\rangle_N E'(\partial\mu_2^\varnothing)E'(\partial\nu_2^\varnothing)E'(\partial\mu_3^{yz})E'(\partial\nu_3^\varnothing) \\ &\le \langle\sigma_{wx}\rangle_K E(\partial\psi_2^\varnothing\partial\psi_3^{yz} \mid C_z), \end{aligned} \tag{4.27}$$

where E' denotes expectation conditional on C_z, and we have used Lemma 4.4. Returning to (4.25)–(4.26),

$$\begin{aligned} &E\big(\partial\psi_2^\varnothing\partial\psi_3^{wxyz}\cdot 1\{w \nleftrightarrow z\}\cdot 1\{w \leftrightarrow x\}\big) \\ &\quad \le \langle\sigma_{wx}\rangle E(\partial\psi_2^\varnothing\partial\psi_3^{yz}\cdot 1\{w \nleftrightarrow z\}). \end{aligned}$$

The other term in (4.25) satisfies the same inequality with x and y interchanged. Inequality (4.24) follows on applying the switching lemma to the right sides of these two last inequalities, and adding them.

The concavity of $\langle\sigma_x\rangle$ follows from the fact that, if

$$T = \sum_{k=1}^n a_k 1_{A_k} \tag{4.28}$$

is a step function on K with $a_k \ge 0$ for all k, and $\gamma(\cdot) = \gamma_1(\cdot) + \alpha T(\cdot)$, then

$$\frac{\partial^2}{\partial\alpha^2}\langle\sigma_x\rangle = \sum_{k,l=1}^n a_k a_l \iint_{A_k\times A_l} dy\, dz\, \langle\sigma_x;\sigma_y;\sigma_z\rangle \le 0. \tag{4.29}$$

Thus, the claim holds whenever $\gamma_2 - \gamma_1$ is a step function. The general claim follows by approximating $\gamma_2 - \gamma_1$ by step functions, and applying the dominated convergence theorem. □

For the next lemma we assume for simplicity that $\gamma \equiv 0$ (although similar results can easily be proved for $\gamma \not\equiv 0$). We let $\bar\delta \in \mathbb{R}$ be an upper bound for δ, thus $\delta(x) \le \bar\delta < \infty$ for all $x \in K$. Let $a, b \in K$ be two distinct points. A closed set $T \subseteq K$ is said to *separate* a from b if every lattice path from a to b (whatever the set of bridges) intersects T. Moreover, if $\varepsilon > 0$ and T separates a from b, we say that T is an *ε-fat separating set* if every point in T lies in a closed sub-interval of T of length at least ε.

Lemma 4.7 (Simon inequality) *Let $\gamma \equiv 0$. If $\varepsilon > 0$ and T is an ε-fat separating set for $a, b \in K$,*

$$\langle\sigma_a\sigma_b\rangle \le \frac{1}{\varepsilon}\exp(8\varepsilon\bar\delta)\int_T \langle\sigma_a\sigma_x\rangle\langle\sigma_x\sigma_b\rangle\, dx. \tag{4.30}$$

Proof By Theorems 3.1 and 4.2,

$$\langle\sigma_a\sigma_x\rangle\langle\sigma_x\sigma_b\rangle = \frac{1}{Z^2}E(\partial\psi_1^\varnothing\partial\psi_2^{ab}\cdot 1\{a \leftrightarrow x\}), \tag{4.31}$$

and, by Fubini's theorem,

$$\int_T \langle\sigma_a\sigma_x\rangle\langle\sigma_x\sigma_b\rangle\, dx = \frac{1}{Z^2}E(\partial\psi_1^\varnothing\partial\psi_2^{ab}\cdot|\widehat{T}|), \tag{4.32}$$

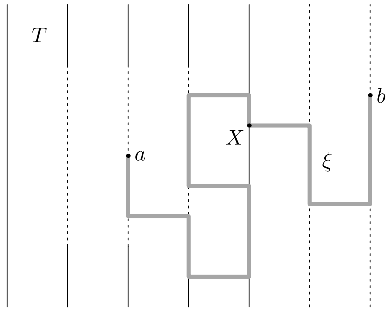

Fig. 6 The Simon inequality. The separating set T is drawn with *solid black lines*, and the backbone ξ with a *grey line*

where $\widehat{T} = \{x \in T : a \leftrightarrow x\}$ and $|\cdot|$ denotes Lebesgue measure. Since $\gamma \equiv 0$, the backbone $\xi = \xi(\psi_2^{ab})$ consists of a single (lattice-) path from a to b passing through T. Let U denote the set of points in K that are separated from b by T, and let X be the point at which ξ exits U for the first time. Since T is assumed closed, $X \in T$. See Fig. 6.

For $x \in T$, let A_x be the event that there is no element of Δ within the interval of length 2ε centred at x. Thus, $P(A_x) \geq \exp(-8\varepsilon\bar{\delta})$. On the event A_X, we have that $|\widehat{T}| \geq \varepsilon$, whence

$$\begin{aligned} E(\partial\psi_1^{\varnothing}\partial\psi_2^{ab} \cdot |\widehat{T}|) &\geq E(\partial\psi_1^{\varnothing}\partial\psi_2^{ab} \cdot |\widehat{T}| \cdot 1\{A_X\}) \\ &\geq \varepsilon E(\partial\psi_1^{\varnothing}\partial\psi_2^{ab} \cdot 1\{A_X\}). \end{aligned} \tag{4.33}$$

Conditional on X, the event A_X is independent of $\psi_1^{\varnothing}$ and ψ_2^{ab}, so that

$$E(\partial\psi_1^{\varnothing}\partial\psi_2^{ab} \cdot |\widehat{T}|) \geq \varepsilon \exp(-8\varepsilon\bar{\delta}) E(\partial\psi_1^{\varnothing}\partial\psi_2^{ab}), \tag{4.34}$$

and the proof is complete. □

Just as for the classical Ising model, only a small amount of extra work is required to obtain the following improvement of Lemma 4.7.

Lemma 4.8 (Lieb inequality) *Under the assumptions of Lemma 4.7,*

$$\langle\sigma_a\sigma_b\rangle \leq \frac{1}{\varepsilon}\exp(8\varepsilon\bar{\delta}) \int_T \langle\sigma_a\sigma_x\rangle_U \, \langle\sigma_x\sigma_b\rangle \, dx, \tag{4.35}$$

where $\langle\cdot\rangle_U$ denotes expectation with respect to the measure restricted to U.

Proof Let $x \in T$, let $\overline{\psi}_1^{ax}$ denote a colouring on the restricted region U, and let ψ_2^{xb} denote a colouring on the full region K as before. We claim that

$$E(\partial\overline{\psi}_1^{ax}\partial\psi_2^{xb}) = E\big(\partial\overline{\psi}_1^{\varnothing}\partial\psi_2^{ab} \cdot 1\{a \leftrightarrow x \text{ in } U\}\big). \tag{4.36}$$

The use of the letter E is an abuse of notation, since the $\overline{\psi}$ are colourings of U only.

Equation (4.36) may be established using a slight variation in the proof of the switching lemma. We follow the proof of that lemma, first conditioning on the set Q of all bridges and ghost-bonds in the two colourings taken together, and then allocating them to the colourings Q_1 and Q_2, uniformly at random. We then order the paths π of Q from a to x, and add the earliest open path to both Q_1 and Q_2 ‘modulo 2’. There are two differences here: firstly, any element of Q that is not contained in U will be allocated to Q_2, and secondly, we only consider paths π that lie inside U. Subject to these two changes, we follow the argument of the switching lemma to arrive at (4.36).

Integrating (4.36) over $x \in T$,

$$\int_T \langle \sigma_a \sigma_x \rangle_U \, \langle \sigma_x \sigma_b \rangle \, dx = \frac{1}{Z_U Z} E(\partial \overline{\psi}_1^{\varnothing} \partial \psi_2^{ab} \cdot |\widehat{T}|), \tag{4.37}$$

where this time $\widehat{T} = \{x \in T : a \leftrightarrow x \text{ in } U\}$. The proof is completed as in (4.33)–(4.34). □

For the next lemma we specialize to the situation that is the main focus of this article, namely the following. Similar results are valid for other lattices and for summable translation-invariant interactions.

Assumption 4.9

– The graph $L = [-n, n]^d \subseteq \mathbb{Z}^d$ where $d \geq 1$, with periodic boundary condition.
– The parameters λ, δ, γ are non-negative constants.
– The set $K_v = \mathbb{S}$ for every $v \in V$.

Under the periodic boundary condition, two vertices of L are joined by an edge whenever there exists $i \in \{1, 2, \ldots, d\}$ such that their i-coordinates differ by exactly $2n$, and all other coordinates are equal.

Under Assumption 4.9, the process is invariant under automorphisms of L and, furthermore, the quantity $\langle \sigma_x \rangle$ does not depend on the choice of x. Let 0 denote some fixed but arbitrary point of K, and let $M = M(\lambda, \delta, \gamma) = \langle \sigma_0 \rangle$ denote the common value of the $\langle \sigma_x \rangle$.

For $x, y \in K$, we write $x \sim y$ if $x = (u, t)$ and $y = (v, t)$ for some $t \geq 0$ and u, v adjacent in L. We write $\{x \overset{z}{\leftrightarrow} y\}$ for the complement of the event that there exists an open path from x to y not containing z. Thus, $x \overset{z}{\leftrightarrow} y$ if: either $x \nleftrightarrow y$, or $x \leftrightarrow y$ and every open path from x to y passes through z.

Theorem 4.10 *Under Assumption 4.9, the following hold.*

$$\frac{\partial M}{\partial \gamma} = \frac{1}{Z^2} \int_K dx \, E\big(\partial \psi_1^{0x} \partial \psi_2^{\varnothing} \cdot 1\{0 \nleftrightarrow \Gamma\}\big) \leq \frac{M}{\gamma}, \tag{4.38}$$

$$\frac{\partial M}{\partial \lambda} = \frac{1}{2Z^2} \int_K dx \sum_{y \sim x} E\big(\partial \psi_1^{0xy\Gamma} \partial \psi_2^{\varnothing} \cdot 1\{0 \nleftrightarrow \Gamma\}\big) \leq 2dM \frac{\partial M}{\partial \gamma}, \tag{4.39}$$

$$-\frac{\partial M}{\partial \delta} = \frac{2}{Z^2} \int_K dx \, E\big(\partial \psi_1^{0\Gamma} \partial \psi_2^{\varnothing} \cdot 1\{0 \overset{x}{\leftrightarrow} \Gamma\}\big) \leq \frac{2M}{1 - M^2} \frac{\partial M}{\partial \gamma}. \tag{4.40}$$

Proof With the exception of (4.40), the proofs mimic those of [7] for the classical Ising model, and are therefore omitted. See [12].

Here is the proof of (4.40). Let $|\cdot|$ denote Lebesgue measure as usual. By differentiating

$$M = \frac{E(\partial \psi^{0\Gamma})}{E(\partial \psi^{\varnothing})} = \frac{E(\exp(2\delta |\mathrm{ev}(\psi^{0\Gamma})|))}{E(\exp(2\delta |\mathrm{ev}(\psi^{\varnothing})|))} \tag{4.41}$$

with respect to δ, we obtain that

$$\begin{aligned}\frac{\partial M}{\partial \delta} &= \frac{2}{Z^2} E\big(\partial\psi_1^{0\Gamma}\partial\psi_2^{\varnothing}\cdot\big[|\mathrm{ev}(\psi_1^{0\Gamma})|-|\mathrm{ev}(\psi_2^{\varnothing})|\big]\big)\\ &= \frac{2}{Z^2}\int dx\, E\big(\partial\psi_1^{0\Gamma}\partial\psi_2^{\varnothing}\cdot\big[1\{x\in\mathrm{odd}(\psi_2^{\varnothing})\}-1\{x\in\mathrm{odd}(\psi_1^{0\Gamma})\}\big]\big).\end{aligned} \tag{4.42}$$

Consider the integrand in (4.42). Since $\psi_2^{\varnothing}$ has no sources, all odd routes in $\psi_2^{\varnothing}$ are necessarily cycles. If $x\in\mathrm{odd}(\psi_2^{\varnothing})$, then x lies in an odd cycle. We may assume that x is not the endpoint of a bridge, since this event has probability 0. It follows that, on the event $\{0\leftrightarrow\Gamma\}$, there exists an open path from 0 to Γ that avoids x (since any path can be re-routed around the odd cycle of $\psi_2^{\varnothing}$ containing x). Therefore, the event $\{0\overset{x}{\leftrightarrow}\Gamma\}$ does not occur, and hence

$$\begin{aligned}&E\big(\partial\psi_1^{0\Gamma}\partial\psi_2^{\varnothing}\cdot 1\{x\in\mathrm{odd}(\psi_2^{\varnothing})\}\big)\\ &\quad= E\big(\partial\psi_1^{0\Gamma}\partial\psi_2^{\varnothing}\cdot 1\{x\in\mathrm{odd}(\psi_2^{\varnothing})\}\cdot 1\{0\overset{x}{\leftrightarrow}\Gamma\}^{\mathrm{c}}\big).\end{aligned} \tag{4.43}$$

If $\partial\psi_1^{0\Gamma}\neq 0$ and $0\overset{x}{\leftrightarrow}\Gamma$, then necessarily $x\in\mathrm{odd}(\psi_1^{0\Gamma})$. Hence,

$$\begin{aligned}&E\big(\partial\psi_1^{0\Gamma}\partial\psi_2^{\varnothing}\cdot 1\{x\in\mathrm{odd}(\psi_1^{0\Gamma})\}\big)\\ &\quad= E\big(\partial\psi_1^{0\Gamma}\partial\psi_2^{\varnothing}\cdot 1\{x\in\mathrm{odd}(\psi_1^{0\Gamma})\}\cdot 1\{0\overset{x}{\leftrightarrow}\Gamma\}^{\mathrm{c}}\big)\\ &\qquad+ E\big(\partial\psi_1^{0\Gamma}\partial\psi_2^{\varnothing}\cdot 1\{0\overset{x}{\leftrightarrow}\Gamma\}\big).\end{aligned} \tag{4.44}$$

We wish to switch the sources 0Γ from ψ_1 to ψ_2 in the right side of (4.44). For this we need to adapt some details of the proof of the switching lemma to this situation. The first step in the proof of that lemma was to condition on the union Q of the bridges and ghost-bonds of the two colourings; then, the paths from 0 to Γ in Q were listed in a fixed *but arbitrary* order. We are free to choose this ordering in such a way that paths not containing x have precedence, and we assume henceforth that the ordering is thus chosen. The next step is to find the earliest open path π, and 'add π modulo 2' to both $\psi_1^{0\Gamma}$ and $\psi_2^{\varnothing}$. On the event $\{0\overset{x}{\leftrightarrow}\Gamma\}^{\mathrm{c}}$, this earliest path π does not contain x, by our choice of ordering. Hence, in the new colouring $\psi_1^{\varnothing}$, x continues to lie in an 'odd' interval (recall that, outside π, the colourings are unchanged by the switching procedure). Therefore,

$$\begin{aligned}&E\big(\partial\psi_1^{0\Gamma}\partial\psi_2^{\varnothing}\cdot 1\{x\in\mathrm{odd}(\psi_1^{0\Gamma})\}\cdot 1\{0\overset{x}{\leftrightarrow}\Gamma\}^{\mathrm{c}}\big)\\ &\quad= E\big(\partial\psi_1^{\varnothing}\partial\psi_2^{0\Gamma}\cdot 1\{x\in\mathrm{odd}(\psi_1^{\varnothing})\}\cdot 1\{0\overset{x}{\leftrightarrow}\Gamma\}^{\mathrm{c}}\big).\end{aligned} \tag{4.45}$$

Relabelling, putting this into (4.44), and subtracting (4.44) from (4.43), we obtain

$$\frac{\partial M}{\partial\delta} = -\frac{2}{Z^2}\int dx\, E\big(\partial\psi_1^{0\Gamma}\partial\psi_2^{\varnothing}\cdot 1\{0\overset{x}{\leftrightarrow}\Gamma\}\big) \tag{4.46}$$

as required.

Turning to the inequality, let C_z^x denote the set of points that can be reached from z along open paths *not containing* x. When calculating the conditional expectation of $\partial\psi_1^{0\Gamma}\partial\psi_2^{\varnothing}\cdot 1\{0\overset{x}{\leftrightarrow}\Gamma\}$ given C_0^x, as in the proof of the GHS inequality, we find that $\psi_1^{0\Gamma}$ is a combination of two independent colourings, one inside C_0^x with sources $0x$, and one outside C_0^x with sources $x\Gamma$. As in (4.27), using Lemma 4.4 as there,

$$\begin{aligned} E\big(\partial\psi_1^{0\Gamma}\partial\psi_2^{\varnothing}\cdot 1\{0 \overset{x}{\leftrightarrow} \Gamma\}\big) &= E\big(\partial\psi_1^{0x}\partial\psi_2^{\varnothing}\langle\sigma_x\rangle_{K\setminus C_0^x}\cdot 1\{0 \overset{x}{\leftrightarrow} \Gamma\}\big) \\ &\le M\cdot E\big(\partial\psi_1^{0x}\partial\psi_2^{\varnothing}\cdot 1\{0 \overset{x}{\leftrightarrow} \Gamma\}\big). \end{aligned} \tag{4.47}$$

We split the expectation on the right side according to whether or not $x \leftrightarrow \Gamma$. Clearly,

$$E\big(\partial\psi_1^{0x}\partial\psi_2^{\varnothing}\cdot 1\{0 \overset{x}{\leftrightarrow} \Gamma\}\cdot 1\{x \not\leftrightarrow \Gamma\}\big) \le E\big(\partial\psi_1^{0x}\partial\psi_2^{\varnothing}\cdot 1\{x \not\leftrightarrow \Gamma\}\big). \tag{4.48}$$

By Switching Lemma 4.2, the other term satisfies

$$E\big(\partial\psi_1^{0x}\partial\psi_2^{\varnothing}\cdot 1\{0 \overset{x}{\leftrightarrow} \Gamma\}\cdot 1\{x \leftrightarrow \Gamma\}\big) = E\big(\partial\psi_1^{0\Gamma}\partial\psi_2^{x\Gamma}\cdot 1\{0 \overset{x}{\leftrightarrow} \Gamma\}\big). \tag{4.49}$$

We again condition on a cluster, this time C_Γ^x, to obtain as in (4.47) that

$$E\big(\partial\psi_1^{0\Gamma}\partial\psi_2^{x\Gamma}\cdot 1\{0 \overset{x}{\leftrightarrow} \Gamma\}\big) \le M\cdot E\big(\partial\psi_1^{0\Gamma}\partial\psi_2^{\varnothing}\cdot 1\{0 \overset{x}{\leftrightarrow} \Gamma\}\big). \tag{4.50}$$

Combining (4.47), (4.48), (4.50) with (4.46), we obtain by (4.38) that

$$-\frac{\partial M}{\partial\delta} \le 2M\frac{\partial M}{\partial\gamma} + M^2\Big(-\frac{\partial M}{\partial\delta}\Big), \tag{4.51}$$

as required. □

5 Proof of Theorem 2.2

In this section we will prove the differential inequality (2.9) which, in combination with the inequalities of the previous section, will yield information about the critical behaviour of the space–time Ising model. The proof proceeds roughly as follows. In the random-parity representation of $M = \langle\sigma_0\rangle$, there is a backbone from 0 to Γ (that is, to some point $g \in G$). We introduce two new sourceless configurations; depending on how the backbone interacts with these configurations, the switching lemma allows a decomposition into a combination of other configurations which, via Theorem 4.10, may be expressed in terms of derivatives of the magnetization.

Throughout this section we work under Assumption 4.9, that is, *we work with a translation-invariant nearest-neighbour model on a cube in the d-dimensional lattice*, while noting that our conclusions are valid for more general interactions with similar symmetries. The arguments in this section borrow heavily from [7]. As in Theorem 4.10, the main novelty in the proof concerns connectivity in the 'vertical' direction (the term R_v in (5.2)–(5.3) below).

By Theorem 3.1,

$$M = \frac{1}{Z}E(\partial\psi_1^{0\Gamma}) = \frac{1}{Z^3}E(\partial\psi_1^{0\Gamma}\partial\psi_2^{\varnothing}\partial\psi_3^{\varnothing}). \tag{5.1}$$

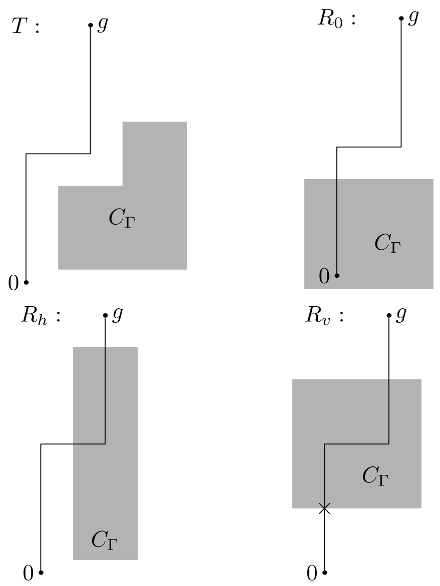

Fig. 7 Illustrations of the four possibilities for $\xi \cap C_\Gamma$. Ghost-bonds in $\psi^{0\Gamma}$ are labelled g. The backbone ξ is drawn as a *solid black line*, and C_Γ as a *grey rectangle*

We shall consider the backbone $\xi = \xi(\psi_1^{0\Gamma})$ and the open cluster C_Γ of Γ in $(\psi_2^\varnothing, \psi_3^\varnothing, \Delta)$. All connectivities will refer to the triple $(\psi_2^\varnothing, \psi_3^\varnothing, \Delta)$. Note that ξ consists of a single path with endpoints 0 and Γ. There are four possibilities, illustrated in Fig. 7, for the way in which ξ, viewed as a directed path from 0 to Γ, interacts with C_Γ:

(i) $\xi \cap C_\Gamma$ is empty,
(ii) $0 \in \xi \cap C_\Gamma$,
(iii) $0 \notin \xi \cap C_\Gamma$, and ξ first meets C_Γ immediately after a bridge,
(iv) $0 \notin \xi \cap C_\Gamma$, and ξ first meets C_Γ at a cut, which necessarily belongs to $\mathrm{ev}(\psi_2^\varnothing) \cap \mathrm{ev}(\psi_3^\varnothing)$.

Thus,

$$M = T + R_0 + R_h + R_v, \tag{5.2}$$

where

$$\begin{aligned}
T &= \frac{1}{Z^3} E\big(\partial\psi_1^{0\Gamma}\partial\psi_2^\varnothing\partial\psi_3^\varnothing \cdot 1\{\xi \cap C_\Gamma = \varnothing\}\big),\\
R_0 &= \frac{1}{Z^3} E\big(\partial\psi_1^{0\Gamma}\partial\psi_2^\varnothing\partial\psi_3^\varnothing \cdot 1\{0 \leftrightarrow \Gamma\}\big),\\
R_h &= \frac{1}{Z^3} E\big(\partial\psi_1^{0\Gamma}\partial\psi_2^\varnothing\partial\psi_3^\varnothing \cdot 1\{\text{first point of } \xi \cap C_\Gamma \text{ is at a bridge of } \xi\}\big),\\
R_v &= \frac{1}{Z^3} E\big(\partial\psi_1^{0\Gamma}\partial\psi_2^\varnothing\partial\psi_3^\varnothing \cdot 1\{\text{first point of } \xi \cap C_\Gamma \text{ is a cut}\}\big).
\end{aligned} \tag{5.3}$$

We will bound each of these terms in turn.

By the switching lemma,

$$\begin{aligned} R_0 &= \frac{1}{Z^3} E\big(\partial\psi_1^{0\Gamma}\partial\psi_2^{\varnothing}\partial\psi_3^{\varnothing} \cdot 1\{0 \leftrightarrow \Gamma\}\big) \\ &= \frac{1}{Z^3} E\big(\partial\psi_1^{0\Gamma}\partial\psi_2^{0\Gamma}\partial\psi_3^{0\Gamma}\big) = M^3. \end{aligned} \tag{5.4}$$

Next, we bound T. The letter ξ will always denote the backbone of the first colouring ψ_1, with corresponding sources. Let X denote the location of the ghost-bond that ends ξ. By conditioning on X,

$$\begin{aligned} T &= \frac{1}{Z^3} \int P(X \in dx)\, E\big(\partial\psi_1^{0\Gamma}\partial\psi_2^{\varnothing}\partial\psi_3^{\varnothing} \cdot 1\{\xi \cap C_\Gamma = \varnothing\} \,\big|\, X = x\big) \\ &\le \frac{\gamma}{Z^3} \int dx\, E\big(\partial\psi_1^{0x}\partial\psi_2^{\varnothing}\partial\psi_3^{\varnothing} \cdot 1\{\xi \cap C_\Gamma = \varnothing\}\big). \end{aligned} \tag{5.5}$$

We study the last expectation by conditioning on C_Γ and bringing one of the factors $1/Z$ inside. By (3.28)–(3.29) and conditional expectation,

$$\begin{aligned} &\frac{1}{Z} E\big(\partial\psi_1^{0x} \cdot 1\{\xi \cap C_\Gamma = \varnothing\} \,\big|\, C_\Gamma\big) \\ &\quad = E\big(Z^{-1} E(\partial\psi_1^{0x} \mid \xi, C_\Gamma) 1\{\xi \cap C_\Gamma = \varnothing\} \,\big|\, C_\Gamma\big) \\ &\quad = E\big(w^{0x}(\xi) \cdot 1\{\xi \cap C_\Gamma = \varnothing\} \,\big|\, C_\Gamma\big). \end{aligned} \tag{5.6}$$

By Lemma 4.5,

$$w^{0x}(\xi) \le 2^{r(\xi) - r'(\xi)} w^{0x}_{K \setminus C_\Gamma}(\xi) \quad \text{on } \{\xi \cap C_\Gamma = \varnothing\}, \tag{5.7}$$

where

$$r(\xi) = r(\xi, K), \qquad r'(\xi) = r(\xi, K \setminus C_\Gamma).$$

Using (3.32) and (3.30), we have

$$\begin{aligned} &E\big(w^{0x}(\xi) \cdot 1\{\xi \cap C_\Gamma = \varnothing\} \,\big|\, C_\Gamma\big) \\ &\quad \le E\big(2^{r(\xi) - r'(\xi)} w^{0x}_{K \setminus C_\Gamma}(\xi) \cdot 1\{\xi \cap C_\Gamma = \varnothing\} \,\big|\, C_\Gamma\big) \\ &\quad \le \langle \sigma_0 \sigma_x \rangle_{K \setminus C_\Gamma}. \end{aligned} \tag{5.8}$$

The last equation merits explanation. Recall that $\xi = \xi(\psi_1^{0x})$, and assume $\xi \cap C_\Gamma = \varnothing$. Apart from the randomization that takes place when ψ_1^{0x} is one of several valid colourings, the law of ξ, $P(\xi \in d\nu)$, is a function of the positions of bridges and ghost-bonds along ν only, that is, the existence of bridges where needed, and the non-existence of ghost-bonds along ν. By (5.7) and Lemma 4.5, with $\Xi_{K \setminus C} := \{\nu \in \Xi : \nu \cap C = \varnothing\}$ and P the law of ξ,

$$\begin{aligned} &E\big(w^{0x}(\xi) \cdot 1\{\xi \cap C_\Gamma = \varnothing\} \,\big|\, C_\Gamma\big) \\ &\quad = \int_{\Xi_{K \setminus C_\Gamma}} w^{0x}(\nu)\, P(d\nu) \\ &\quad \le \int_{\Xi_{K \setminus C_\Gamma}} 2^{r(\nu) - r'(\nu)} w^{0x}_{K \setminus C_\Gamma}(\nu) \left(\tfrac{1}{2}\right)^{r(\nu)} \mu(d\nu) \end{aligned}$$

for some measure μ, where the factor $(\frac{1}{2})^{r(\nu)}$ arises from the possible existence of more than one valid colouring. Now, μ is a measure on paths which, by the remark above, depends only locally on ν, in the sense that $\mu(d\nu)$ depends only on the bridge- and ghost-bond configurations along ν. In particular, the same measure μ governs also the law of the backbone in the *smaller* region $K \setminus C_\Gamma$. More explicitly, by (3.30) with $P_{K\setminus C_\Gamma}$ the law of the backbone of the colouring $\psi^{0x}_{K\setminus C_\Gamma}$ defined on $K \setminus C_\Gamma$, we have

$$\begin{aligned}\langle\sigma_0\sigma_x\rangle_{K\setminus C_\Gamma} &= \int_{\Xi_{K\setminus C_\Gamma}} w^{0x}_{K\setminus C_\Gamma}(\nu)\, P_{K\setminus C_\Gamma}(d\nu)\\ &= \int_{\Xi_{K\setminus C_\Gamma}} w^{0x}_{K\setminus C_\Gamma}(\nu)\left(\tfrac{1}{2}\right)^{r'(\nu)} \mu(d\nu).\end{aligned}$$

Thus (5.8) follows.

Therefore, by (5.5)–(5.8),

$$\begin{aligned}T &\le \frac{\gamma}{Z^2}\int dx\, E\big(\partial\psi_2^{\varnothing}\partial\psi_3^{\varnothing}\langle\sigma_0\sigma_x\rangle_{K\setminus C_\Gamma}\cdot 1\{0\nleftrightarrow\Gamma\}\big)\\ &= \gamma\int dx\,\frac{1}{Z^2}E\big(\partial\psi_2^{0x}\partial\psi_3^{\varnothing}\cdot 1\{0\nleftrightarrow\Gamma\}\big)\\ &= \gamma\frac{\partial M}{\partial\gamma},\end{aligned}$$

by ‘conditioning on the cluster’ C_Γ and Theorem 4.10.

Next, we bound R_h. Suppose that the bridge bringing ξ into C_Γ has endpoints X and Y, where we take X to be the endpoint not in C_Γ. When the bridge XY is removed, the backbone ξ consists of two paths: $\zeta^1 : 0 \to X$ and $\zeta^2 : Y \to \Gamma$. Therefore,

$$\begin{aligned}R_h &= \frac{1}{Z^3}\int P(X\in dx)\, E\big(\partial\psi_1^{0\Gamma}\partial\psi_2^{\varnothing}\partial\psi_3^{\varnothing}\,\big|\, X = x\big)\\ &\le \frac{\lambda}{Z^3}\int dx\sum_{y\sim x} E\big(\partial\psi_1^{0xy\Gamma}\partial\psi_2^{\varnothing}\partial\psi_3^{\varnothing}\cdot 1\{0\nleftrightarrow\Gamma,\ y\leftrightarrow\Gamma\}\cdot 1\{J_\xi\}\big),\end{aligned}$$

where $\xi = \xi(\psi_1^{0xy\Gamma})$ and

$$J_\xi = \big\{\xi = \zeta^1\circ\zeta^2,\ \zeta^1 : 0\to x,\ \zeta^2 : y\to\Gamma,\ \zeta^1\cap C_\Gamma = \varnothing\big\}.$$

As in (5.6),

$$R_h \le \frac{\lambda}{Z^2}\int dx\sum_{y\sim x} E\big(\partial\psi_2^{\varnothing}\partial\psi_3^{\varnothing}\cdot 1\{0\nleftrightarrow\Gamma,\ y\leftrightarrow\Gamma\}\cdot w^{0xy\Gamma}(\xi)\cdot 1\{J_\xi\}\big). \tag{5.9}$$

By Lemmas 3.3(a) and 4.5, on the event J_ξ,

$$\begin{aligned}w^{0xy\Gamma}(\xi) &= w^{0x}(\zeta^1)\, w^{y\Gamma}_{K\setminus\zeta^1}(\zeta^2)\\ &\le 2^{r-r'} w^{0x}_{K\setminus C_\Gamma}(\zeta^1)\, w^{y\Gamma}_{K\setminus\zeta^1}(\zeta^2),\end{aligned}$$

where $r = r(\zeta^1, K)$ and $r' = r(\zeta^1, K \setminus C_\Gamma)$. By Lemma 4.4 and the reasoning after (5.8),

$$\begin{aligned}E\big(w^{0xy\Gamma}(\xi)\cdot 1\{J_\xi\}\,\big|\,\zeta^1, C_\Gamma\big) &\le 2^{r-r'} w^{0x}_{K\setminus C_\Gamma}(\zeta^1)\cdot\langle\sigma_y\rangle_{K\setminus\zeta^1}\\ &\le M\cdot 2^{r-r'} w^{0x}_{K\setminus C_\Gamma}(\zeta^1),\end{aligned}$$

so that, similarly,

$$E\big(w^{0xy\Gamma}(\xi)\cdot 1\{J_\xi\}\,\big|\,C_\Gamma\big) \le M\cdot\langle\sigma_0\sigma_x\rangle_{K\setminus C_\Gamma}. \tag{5.10}$$

We substitute into the summand in (5.9), using the switching lemma, conditioning on the cluster C_Γ, and the bound $\langle\sigma_y\rangle_{C_\Gamma} \le M$, to obtain the upper bound

$$\begin{aligned}&M\cdot E\big(\partial\psi_2^{\varnothing}\partial\psi_3^{\varnothing}\cdot 1\{0\nleftrightarrow\Gamma,\ y\leftrightarrow\Gamma\}\cdot\langle\sigma_0\sigma_x\rangle_{K\setminus C_\Gamma}\big)\\ &\quad= M\cdot E\big(\partial\psi_2^{y\Gamma}\partial\psi_3^{y\Gamma}\cdot 1\{0\nleftrightarrow\Gamma\}\cdot\langle\sigma_0\sigma_x\rangle_{K\setminus C_\Gamma}\big)\\ &\quad= M\cdot E\big(\partial\psi_2^{0xy\Gamma}\partial\psi_3^{\varnothing}\langle\sigma_y\rangle_{C_\Gamma}\cdot 1\{0\nleftrightarrow\Gamma\}\big)\\ &\quad\le M^2\cdot E\big(\partial\psi_2^{0xy\Gamma}\partial\psi_3^{\varnothing}\cdot 1\{0\nleftrightarrow\Gamma\}\big).\end{aligned}$$

Hence, by (4.39),

$$\begin{aligned}R_h &\le \lambda M^2\frac{1}{Z^2}\int dx\sum_{y\sim x}E\big(\partial\psi_2^{0xy\Gamma}\partial\psi_3^{\varnothing}1\{0\nleftrightarrow\Gamma\}\big)\\ &= 2\lambda M^2\frac{\partial M}{\partial\lambda}.\end{aligned}$$

Finally, we bound R_v. Let $X\in\Delta\cap\mathrm{ev}(\psi_2^{\varnothing})\cap\mathrm{ev}(\psi_3^{\varnothing})$ be the first point of ξ in C_Γ. In a manner similar to that used for R_h at (5.9) above, and by cutting the backbone ξ at the point x,

$$R_v \le \frac{1}{Z^2}\int P(X\in dx)\,E\big(\partial\psi_2^{\varnothing}\partial\psi_3^{\varnothing}\cdot 1\{0\nleftrightarrow\Gamma,\ x\leftrightarrow\Gamma\}\cdot w^{0\Gamma}(\xi)\cdot 1\{J_\xi\}\big), \tag{5.11}$$

where

$$J_\xi = 1\big\{\xi = \overline{\zeta}^1\circ\overline{\zeta}^2,\ \overline{\zeta}^1: 0\to x,\ \overline{\zeta}^2: x\to\Gamma,\ \zeta^1\cap C_\Gamma=\varnothing\big\}.$$

As in (5.10),

$$\begin{aligned}E(w^{0\Gamma}(\xi)\cdot 1\{J_\xi\}\mid C_\Gamma) &= E\big(E(w^{0\Gamma}(\xi)\cdot 1\{J_\xi\}\mid\overline{\zeta}^1, C_\Gamma)\,\big|\,C_\Gamma\big)\\ &\le E\big(\langle\sigma_0\sigma_x\rangle_{K\setminus C_\Gamma}\cdot\langle\sigma_x\rangle_{K\setminus\zeta^1}\,\big|\,C_\Gamma\big)\\ &\le \langle\sigma_0\sigma_x\rangle_{K\setminus C_\Gamma}\cdot M.\end{aligned}$$

By (5.11) therefore,

$$R_v \le M\frac{1}{Z^2}\int P(X\in dx)\,E\big(\partial\psi_2^{\varnothing}\partial\psi_3^{\varnothing}\cdot 1\{0\nleftrightarrow\Gamma,\ x\leftrightarrow\Gamma\}\langle\sigma_0\sigma_x\rangle_{K\setminus C_\Gamma}\big).$$

By removing the cut at x, the origin 0 becomes connected to Γ, but only via x. Thus,

$$R_v \le 4\delta M\frac{1}{Z^2}\int dx\,E\big(\partial\psi_2^{\varnothing}\partial\psi_3^{\varnothing}\cdot 1\{0\overset{x}{\leftrightarrow}\Gamma,\ x\leftrightarrow\Gamma\}\langle\sigma_0\sigma_x\rangle_{K\setminus C_\Gamma^x}\big),$$

where C_Γ^x is the set of points reached from Γ along open paths not containing x. By the switching lemma, and conditioning twice on the cluster C_Γ^x,

$$\begin{aligned}
R_v &\le 4\delta M \frac{1}{Z^2} \int dx\, E\big(\partial\psi_2^{x\Gamma} \partial\psi_3^{x\Gamma} \cdot 1\{0 \overset{x}{\leftrightarrow} \Gamma\} \langle \sigma_0 \sigma_x \rangle_{K \setminus C_\Gamma^x}\big) \\
&= 4\delta M \frac{1}{Z^2} \int dx\, E\big(\partial\psi_2^{0\Gamma} \partial\psi_3^{x\Gamma} \cdot 1\{0 \overset{x}{\leftrightarrow} \Gamma\}\big) \\
&= 4\delta M \frac{1}{Z^2} \int dx\, E\big(\partial\psi_2^{0\Gamma} \partial\psi_3^{\varnothing} \cdot 1\{0 \overset{x}{\leftrightarrow} \Gamma\} \langle \sigma_x \rangle_{C_\Gamma^x}\big) \\
&\le 4\delta M^2 \frac{1}{Z^2} \int dx\, E\big(\partial\psi_2^{0\Gamma} \partial\psi_3^{\varnothing} \cdot 1\{0 \overset{x}{\leftrightarrow} \Gamma\}\big) \\
&= -2\delta M^2 \frac{\partial M}{\partial \delta},
\end{aligned}$$

by (4.40), as required.

6 Consequences of the Inequalities

In this section we formulate our principal results, and we indicate how the differential inequalities of Theorems 2.2 and 4.10 may be used to prove them. The arguments used are relatively straightforward adaptations of arguments developed for the classical Ising model, many of which are summarized in [18]. In the interests of brevity, we shall omit many steps, and we hope that readers familiar with the literature will be able to complete the gaps. Full details for the current model may be found in [12]. We work under Assumption 4.9 throughout this section, unless otherwise stated. It is sometimes inconvenient to use periodic boundary conditions, and we revert to the free condition where necessary.

We shall consider the infinite-volume limit as $L \uparrow \mathbb{Z}^d$; the ground state is obtained by letting $\beta \to \infty$ also. Let n be a positive integer, and set $L_n = [-n, n]^d$ with periodic boundary condition. It is convenient (and equivalent) to work instead on the translated space $\Lambda_n^\beta := [-n, n]^d \times [-\frac{1}{2}\beta, \frac{1}{2}\beta]$, and we assume this henceforth. By this device, the limit process as $n, \beta \to \infty$ inhabits $\mathbb{Z}^d \times \mathbb{R}$ rather than $\mathbb{Z}^d \times \mathbb{R}_+$. The symbol β will appear as superscript in the following; the superscript ∞ is to be interpreted as the ground state. Let $0 = (0, 0)$ and

$$M_n^\beta(\lambda, \delta, \gamma) = \langle \sigma_0 \rangle_{L_n}^\beta$$

be the magnetization in Λ_n^β, noting that $M_n^\beta \equiv 0$ when $\gamma = 0$.

By convexity-of-pressure arguments, as developed in [30], the limits

$$M^\beta := \lim_{n\to\infty} M_n^\beta, \qquad M^\infty := \lim_{n\to\infty} \lim_{\beta\to\infty} M_n^\beta, \tag{6.1}$$

exist for Lebesgue-a.e. $\gamma \ge 0$. Moreover, using the GHS inequality as in [34] (which implies the differentiability of the pressure function in γ whenever $\gamma > 0$) and the results of [30], we find that the limits (6.1) exist for all $\gamma > 0$, and are independent of the order of the limits. Note that this argument does not rely on a Lee–Yang theorem. We have that $M^\beta(\lambda, \delta, 0) = 0$.

By a standard re-scaling argument, M^∞ depends only on the ratios λ/δ and γ/δ, and thus we shall set $\delta = 1$, $\rho = \lambda/\delta$, and write

$$M^\beta(\rho, \gamma) = M^\beta(\rho, 1, \gamma), \quad \beta \in (0, \infty],$$

with a similar notation for other functions.

As in [30], when $\gamma > 0$, there exists a unique equilibrium state at (ρ, γ). That is, the limits

$$\langle \sigma_A \rangle^\beta := \lim_{n\to\infty} \langle \sigma_A \rangle_n^\beta, \qquad \langle \sigma_A \rangle^\infty := \lim_{n,\beta\to\infty} \langle \sigma_A \rangle_n^\beta,$$

exist for all A, where $\langle \cdot \rangle_n := \langle \cdot \rangle_{L_n}$, and the limits are independent of the choice of boundary condition. It follows that the infinite-volume probability measure exists (this is a standard exercise using the Skorohod topology, see [10, 19]). A phase transition is manifested by non-uniqueness of the equilibrium state, and this can therefore occur only when $\gamma = 0$. Let $\langle \cdot \rangle_+^\beta$ be the limiting state of $\langle \cdot \rangle^\beta$ as $\gamma \downarrow 0$, and

$$M_+^\beta(\rho) := \lim_{\gamma\downarrow 0} M^\beta(\rho, \gamma).$$

As in [30], there is non-uniqueness at $(\rho, 0)$ if and only if $M_+^\beta(\rho) > 0$, and this motivates the definition

$$\rho_c^\beta := \inf\{\rho > 0 : M_+^\beta(\rho) > 0\}, \tag{6.2}$$

see also (1.3) and (1.5). We shall have need later for the infinite-volume limit $\langle \cdot \rangle^{\mathrm{f},\beta}$, as $n \to \infty$, with *free* boundary condition in the $\mathbb{Z}^d$ direction. Note that

$$\langle \cdot \rangle_{\gamma=0}^{\mathrm{f},\beta} = \langle \cdot \rangle_{\gamma=0}^\beta = \langle \cdot \rangle_+^\beta \quad \text{if } M_+^\beta(\rho) = 0. \tag{6.3}$$

The superscript 'f' shall always indicate this free boundary condition.

Remark 6.1 It is sometimes convenient to work with the random-cluster (or FK) representation of the space–time Ising model, as in [9, 25, 27]. For $\beta \in (0, \infty)$, let $\phi_\rho^{b,\beta}$, $b = 0, 1$, be the $q = 2$ random-cluster measures arising as the limit as $n \to \infty$ of the continuum random-cluster measure on $L_n \times [-\frac{1}{2}\beta, \frac{1}{2}\beta]$ with respectively free/wired boundary condition in the spatial direction. (There are no ghost-bonds, in that $\gamma = 0$.) We define $\phi_\rho^{b,\infty}$ similarly. As discussed in [9, 27], and in [23] for discrete lattices, these limits exist, and are equal for all but countably many values of ρ. (They are presumably equal for all $\rho \neq \rho_c$, using arguments of [8, 13, 23], but we do not pursue this further here.) Furthermore, they are non-decreasing in ρ, and, in particular,

$$\phi_\rho^{1,\beta} \le \phi_{\rho'}^{0,\beta}, \quad \rho < \rho', \tag{6.4}$$

where $\le$ denotes stochastic ordering (see [27]). In the usual manner, for $\beta \in (0, \infty]$,

$$\phi_\rho^{1,\beta}(x \leftrightarrow y) = \langle \sigma_x \sigma_y \rangle_+^\beta, \qquad \phi_\rho^{1,\beta}(0 \leftrightarrow \infty) = M_+(\rho), \tag{6.5}$$

where $\leftrightarrow$ denotes an open connection in the random-cluster model. It may be seen as in [23, Theorems 4.19, 4.23] that the $\phi_\rho^{b,\beta}$ have trivial tail σ-fields, and are thus mixing and ergodic. Therefore, the $\phi_\rho^{b,\beta}$ possess (a.s.) no more than one unbounded cluster, by the

Burton–Keane argument, [14, 23]. By (6.5), the FKG inequality, and the uniqueness of any unbounded cluster,

$$\langle \sigma_x \sigma_y \rangle_+^\beta \ge \phi_\rho^{1,\beta}(x \leftrightarrow \infty)\phi_\rho^{1,\beta}(y \leftrightarrow \infty) = M_+^\beta(\rho)^2. \tag{6.6}$$

Let $\beta \in (0, \infty)$. Using the convexity of Lemma 4.6 as in [18], the derivative $\partial M^\beta / \partial \gamma$ exists for almost every $\gamma \in (0, \infty)$, and, when this holds,

$$\chi_n^\beta(\rho, \gamma) := \frac{\partial M_n^\beta}{\partial \gamma} \to \chi(\rho, \gamma) := \frac{\partial M^\beta}{\partial \gamma} < \infty. \tag{6.7}$$

The corresponding conclusion holds also as $n, \beta \to \infty$. Furthermore, the limits

$$\chi_+^\beta(\rho) := \lim_{\gamma \downarrow 0} \chi^\beta(\rho, \gamma), \quad \beta \in (0, \infty],$$

exist when taken along suitable sequences.

The limit

$$\begin{aligned} \chi^{\mathrm{f},\beta}(\rho, 0) &:= \lim_{n \to \infty} \left(\left. \frac{\partial M_n^{\mathrm{f},\beta}}{\partial \gamma} \right|_{\gamma = 0} \right) \\ &= \lim_{n \to \infty} \int_{\Lambda_n^\beta} \langle \sigma_0 \sigma_x \rangle_{n,\gamma=0}^{\mathrm{f},\beta} \, dx = \int \langle \sigma_0 \sigma_x \rangle_{\gamma=0}^{\mathrm{f},\beta} \, dx \end{aligned} \tag{6.8}$$

exists by monotone convergence, see Lemma 4.4. By Lemma 4.6,

$$\chi_+^\beta(\rho) \ge \chi^{\mathrm{f},\beta}(\rho, 0) \text{ whenever } M_+^\beta(\rho) = 0, \quad \beta \in (0, \infty]. \tag{6.9}$$

Let

$$\rho_{\mathrm{s}}^\beta := \inf\{\rho > 0 : \chi^{\mathrm{f},\beta}(\rho, 0) = \infty\}, \quad \beta \in (0, \infty]. \tag{6.10}$$

By (6.4)–(6.5) and the monotonicity of $\chi^{\mathrm{f},\beta}(\rho, 0)$,

$$\rho_{\mathrm{s}}^\beta \le \rho_{\mathrm{c}}^\beta. \tag{6.11}$$

By the discussion around (6.2)–(6.3), there is a unique equilibrium state when $\gamma = 0$ and $\rho < \rho_{\mathrm{c}}^\beta$. We shall see in Theorem 6.3 that $\chi^{\mathrm{f},\beta}(\rho_{\mathrm{s}}^\beta, 0) = \infty$.

For $x \in \mathbb{Z}^d \times \mathbb{R}$, let $\|x\|$ denote the supremum norm of x.

Theorem 6.2 *Let $\beta \in (0, \infty]$ and $\rho < \rho_{\mathrm{s}}^\beta$. There exists $\alpha = \alpha^\beta(\rho) > 0$ such that*

$$\langle \sigma_0 \sigma_x \rangle_+^\beta \le e^{-\alpha \|x\|}, \quad x \in \mathbb{Z}^d \times \mathbb{R}. \tag{6.12}$$

Proof Fix $\beta \in (0, \infty)$ and $\gamma = 0$, and let $\rho < \rho_{\mathrm{s}}^\beta$, so that (6.3) applies. Therefore,

$$\chi^{\mathrm{f},\beta}(\rho, 0) = \int_{\mathbb{Z}^d \times [-\frac{1}{2}\beta, \frac{1}{2}\beta]} \langle \sigma_0 \sigma_x \rangle^\beta \, dx = \sum_{k \ge 1} \int_{C_k^\beta} \langle \sigma_0 \sigma_x \rangle^\beta \, dx, \tag{6.13}$$

where $C_k^\beta := \Lambda_k^\beta \setminus \Lambda_{k-1}^\beta$. Since $\rho < \rho_{\mathrm{s}}^\beta$, the last summation converges, whence, for sufficiently large k,

$$\int_{C_k^\beta} \langle \sigma_0 \sigma_x \rangle^\beta \, dx < e^{-8}. \tag{6.14}$$

The result now follows in the usual manner by the Simon inequality, Lemma 4.7, with the 1-fat separating sets C_k^β. A similar argument holds when $\beta = \infty$. Further discussion of the method may be found at [23, Corollary 9.38]. □

Let $\beta \in (0, \infty]$, $\gamma = 0$ and define the *mass*

$$m^\beta(\rho) := \liminf_{|x| \to \infty} \left(-\frac{1}{\|x\|} \log \langle \sigma_0 \sigma_x \rangle_\rho^\beta \right). \tag{6.15}$$

By Theorem 6.2 and (6.6),

$$m^\beta(\rho) \begin{cases} > 0, & \text{if } \rho < \rho_{\mathrm{s}}^\beta, \\ = 0, & \text{if } \rho > \rho_{\mathrm{c}}^\beta. \end{cases} \tag{6.16}$$

Theorem 6.3 *Except when $d = 1$ and $\beta < \infty$, $m^\beta(\rho_{\mathrm{s}}^\beta) = 0$ and $\chi^{\mathrm{f},\beta}(\rho_{\mathrm{s}}^\beta, 0) = \infty$.*

Remark 6.4 The manner of the divergence of the susceptibility χ may be studied via the so-called Lebowitz inequalities of [29]. Such inequalities are easily proved for the quantum Ising model using the switching lemma.

Proof Let $d \geq 2$, $\gamma = 0$, and fix $\beta \in (0, \infty)$. We use the Lieb inequality, Lemma 4.8, and the argument of [31, 35], see also [23, Corollary 9.46]. It is necessary and sufficient for $m^\beta(\rho) > 0$ that

$$\int_{C_n^\beta} \langle \sigma_0 \sigma_x \rangle_{n,\rho}^{\mathrm{f},\beta} \, dx < e^{-8} \quad \text{for some } n. \tag{6.17}$$

Necessity holds because the integrand is no greater than $\langle \sigma_0 \sigma_x \rangle^\beta$. Sufficiency follows from Lemma 4.8, as in the proof of Theorem 6.2.

By (2.4),

$$\begin{aligned} \frac{\partial}{\partial \rho} \langle \sigma_0 \sigma_x \rangle_{n,\rho}^{\mathrm{f},\beta} &= \frac{1}{2} \int_{\Lambda_n^\beta} dy \sum_{z \sim y} \langle \sigma_0 \sigma_x ; \sigma_y \sigma_z \rangle_{n,\rho}^{\mathrm{f},\beta} \\ &\leq d\beta (2n+1)^d. \end{aligned}$$

Therefore, if $\rho' > \rho$,

$$\int_{C_n^\beta} \langle \sigma_0 \sigma_x \rangle_{n,\rho'}^{\mathrm{f},\beta} \, dx \leq d[\beta(2n+1)^d]^2 (\rho' - \rho) + \int_{C_n^\beta} \langle \sigma_0 \sigma_x \rangle_{n,\rho}^{\mathrm{f},\beta} \, dx. \tag{6.18}$$

Hence, if (6.17) holds for some ρ, then it holds for ρ' when $\rho' - \rho > 0$ is sufficiently small.

Suppose $m^\beta(\rho_{\mathrm{s}}^\beta) > 0$. Then $m^\beta(\rho') > 0$ for some $\rho' > \rho_{\mathrm{s}}^\beta$, which contradicts $\chi^{\mathrm{f},\beta}(\rho', 0) = \infty$, and the first claim of the theorem follows. A similar argument holds when $d \geq 1$ and $\beta = \infty$. The second claim follows similarly: if $\chi^{\mathrm{f},\beta}(\rho_{\mathrm{s}}^\beta, 0) < \infty$, then (6.17) holds with $\rho = \rho_{\mathrm{s}}^\beta$, whence $m^\beta(\rho') > 0$ and $\chi^{\mathrm{f},\beta}(\rho', 0) < \infty$ for some $\rho' > \rho_{\mathrm{s}}^\beta$, a contradiction. (See also [2].) □

We are now ready to state the main results. The inequalities of Theorems 4.10 and 2.2 may be combined to obtain

$$M_n^\beta \leq (M_n^\beta)^3 + \chi_n^\beta \cdot \left(\gamma + 4d\lambda (M_n^\beta)^3 + 4\delta \frac{(M_n^\beta)^3}{1 - (M_n^\beta)^2} \right). \tag{6.19}$$

Using these inequalities and the facts stated above, it is straightforward to adapt the arguments of [4, Lemmas 4.1, 5.1] (see also [7, 22]) to prove the following. We omit the proofs.

Theorem 6.5 *There are constants $c_1, c_2 > 0$ such that, for $\beta \in (0, \infty]$,*

$$M^\beta(\rho_s, \gamma) \geq c_1 \gamma^{1/3}, \tag{6.20}$$

$$M_+^\beta(\rho) \geq c_2(\rho - \rho_s^\beta)^{1/2}, \tag{6.21}$$

for small positive γ and $\rho - \rho_s^\beta$, respectively.

This is vacuous when $d = 1$ and $\beta < \infty$; see (1.5). The exponents in the above inequalities are presumably sharp in the corresponding mean-field model (see [3, 7] and Remark 6.7). It is standard that a number of important results follow from Theorem 6.5, some of which we state here.

Theorem 6.6 *For $d \geq 1$ and $\beta \in (0, \infty]$, we have that $\rho_c^\beta = \rho_s^\beta$.*

Proof Except when $d = 1$ and $\beta < \infty$, this is immediate from (6.11) and (6.21). In the remaining case, $\rho_c^\beta = \rho_s^\beta = \infty$. □

Remark 6.7 Let $\beta \in (0, \infty]$. Except when $d = 1$ and $\beta < \infty$, one may conjecture the existence of exponents $a = a^\beta(d)$, $b = b^\beta(d)$ such that

$$M_+^\beta(\rho) = (\rho - \rho_c^\beta)^{(1+o(1))a} \quad \text{as } \rho \downarrow \rho_c^\beta, \tag{6.22}$$

$$M^\beta(\rho_c^\beta, \gamma) = \gamma^{(1+o(1))/b} \quad \text{as } \gamma \downarrow 0. \tag{6.23}$$

(We do not exclude the possibility that, when $\beta < \infty$, the values of the exponents depend also on the value of δ.) Theorem 6.5 would then imply that $a \leq \frac{1}{2}$ and $b \geq 3$. In [16, Theorem 3.2] it is proved for the ground-state quantum Curie–Weiss, or mean-field, model that the corresponding $a = \frac{1}{2}$. It may be conjectured (as proved for the classical Ising model in [3]) that the values $a = \frac{1}{2}$ and $b = 3$ are attained for the space–time Ising model on $\mathbb{Z}^d \times [-\frac{1}{2}\beta, \frac{1}{2}\beta]$ for d sufficiently large, that is, when either $\beta < \infty$ and $d \geq 4$, or $\beta = \infty$ and $d \geq 3$.

Finally, a note about (2.14). The random-cluster measure corresponding to the quantum Ising model is *periodic* in both $\mathbb{Z}^d$ and β directions, and this complicates the infinite-volume limit. Since the periodic random-cluster measure dominates the free random-cluster measure, for $\beta \in (0, \infty)$, as in (6.4) and (6.6),

$$\begin{aligned} \liminf_{n\to\infty} \tau_{L_n}^\beta(u, v) \ &\geq\ \langle \sigma_{(u,0)}\sigma_{(v,0)} \rangle_{+,\rho'}^\beta \quad \text{for } \rho' < \rho \\ &\to M_+^\beta(\rho-)^2 \quad \text{as } \rho' \uparrow \rho, \end{aligned}$$

and a similar argument holds in the ground state also.

7 In One Dimension

The space–time version of the quantum Ising model on $\mathbb{Z}$ is two-dimensional, living on $\mathbb{Z} \times \mathbb{R}$. In the light of (1.5), we shall study only the ground state, and we shall suppress the superscript ∞. One may adapt some of the special arguments for two-dimensional models based on planar duality. One consequence is the following.

Theorem 7.1 *Let $d = 1$. Then $\rho_c = 2$, and the transition is of second order in that $M_+(2) = 0$.*

We mention two applications of this theorem. Consider first a 'star-like' graph, comprising finitely many copies of $\mathbb{Z}$, pairs of which may intersect at single points. It is shown in [11], using Theorem 7.1, that the quantum Ising model on such a graph has critical value $\rho_c = 2$.

Secondly, in an account [27] of so-called 'entanglement' in the quantum Ising model on the subset $[-m, m]$ of $\mathbb{Z}$, it was shown that the reduced density matrix ν_m^L of the block $[-L, L]$ satisfies

$$\|\nu_m^L - \nu_n^L\| \le \min\{2, CL^\alpha e^{-cm}\}, \quad 2 \le m < n < \infty,$$

where C and α are constants depending on $\rho = \lambda/\delta$, and $c = c(\rho) > 0$ whenever $\rho < 1$. Using Theorems 6.2 and 7.1, we have that $c(\rho) > 0$ if and only if $\rho < \rho_c = 2$.

Proof We sketch the proof here. It uses the random-cluster (or FK) representation of the equilibrium state $\langle\cdot\rangle_+$, see Remark 6.1. Writing ϕ_ρ^0 (respectively, ϕ_ρ^1) for the free (respectively, wired) $q = 2$ random-cluster measure, we have as in (6.5) that

$$\langle\sigma_x\sigma_y\rangle_+ = \phi_\rho^1(x \leftrightarrow y), \qquad \langle\sigma_x\rangle_+ = \phi_\rho^1(x \leftrightarrow \infty). \tag{7.1}$$

Planar duality is a standard tool in two-dimensional models, and it applies to the random-cluster model on $\mathbb{Z} \times \mathbb{R}$. The details are similar to those in related systems, and the reader is referred to [5, 23, 25] in this regard. There is a standard computation that shows that, in a certain sense that is sensitive to the geometry of the configurations, ϕ_ρ^0 and $\phi_{4/\rho}^1$ form a dual pair of measures.

The argument developed by Zhang for percolation (see [22, 23]) may be adapted to the current setting to obtain that $\rho_c \ge 2$. Roughly speaking, this is as follows. Suppose that $\rho_c < 2$, so that there exists, ϕ_2^0-almost-surely, an unbounded cluster. As in Remark 6.1, for $b = 0, 1$, there exists, ϕ_2^b-almost-surely, a unique unbounded cluster. This implies that both the primal and dual processes at $\rho = 2$ contain unbounded clusters, a possibility that Zhang's construction shows to be contradictory. The argument so far uses no facts proved in the current paper, and it yields that

$$\phi_2^0(0 \leftrightarrow \infty) = 0. \tag{7.2}$$

We show next that $\rho_c \le 2$, following the method developed for percolation to be found in [22, 23]. Suppose that $\rho_c > 2$. By the above duality, one may find a box of side-length n such that: the ϕ_2^1-probability of a crossing of this box is bounded away from 0 uniformly in n. By (7.1) and Theorem 6.2, this probability decays to zero in the manner of $Cne^{-\alpha n}$ as $n \to \infty$, a contradiction.

We show finally that $M_+(2) = 0$ by adapting a simple argument presented by Werner in [37] for the classical Ising model on $\mathbb{Z}^2$. Certain geometrical details are omitted. Let π_2 be the Ising state obtained from a realization of ϕ_2^0 by labelling each open cluster $+1$ with probability $\frac{1}{2}$, and otherwise -1. By (7.2) and a standard argument based on the coupling with the random-cluster measure ϕ_2^0 (see [24, Exercise 8.14]), π_2 is ergodic. The Ising state π_2^+ is obtained similarly from the random-cluster measure ϕ_2^1, with the difference that any infinite cluster is invariably assigned spin $+1$.

We adopt the harmless convention that, for any spin-configuration σ on $\mathbb{Z} \times \mathbb{R}$, the subset labelled $+1$ is closed; the labelling is well-defined except at deaths, and we choose to label a death a with the spin $+1$ if and only if at least one of the intervals abutting a is labelled $+1$.

Let σ be a spin-configuration on $\mathbb{Z} \times \mathbb{R}$. The binary relations $\overset{\pm}{\leftrightarrow}$ are defined as follows. A *path* of $\mathbb{Z} \times \mathbb{R}$ is a self-avoiding path of $\mathbb{R}^2$ that: traverses a finite number of line-segments of $\mathbb{Z} \times \mathbb{R}$, and is permitted to connect them by passing between any two points of the form (u, t), $(u \pm 1, t)$. A path is called a $(+)$path (respectively, $(-)$path) if all its elements are labelled $+1$ (respectively, -1). For $x, y \in \mathbb{Z} \times \mathbb{R}$, we write $x \overset{+}{\leftrightarrow} y$ (respectively, $x \overset{-}{\leftrightarrow} y$) if there exists a $(+)$path (respectively, $(-)$path) with endpoints x, y. Let N^+ (respectively, N^-) be the number of unbounded $+$ (respectively, $-$) Ising clusters with connectivity relation $\overset{+}{\leftrightarrow}$ (respectively, $\overset{-}{\leftrightarrow}$). By the Burton–Keane argument, either $\pi_2(N^+ = 1) = 1$ or $\pi_2(N^+ = 0) = 1$. The former entails also that $\pi_2(N^- = 1) = 1$, and this is impossible by another use of Zhang's argument. Therefore,

$$\pi_2(N^\pm = 0) = 1. \tag{7.3}$$

There is a standard argument for deducing $\pi_2 = \pi_2^+$ from (7.3), of which the idea is roughly as follows. (See [8] or [23, Theorem 5.33] for examples of similar arguments applied to the random-cluster model.) Let $\Lambda_n = [-n, n]^2$, viewed as a subset of $\mathbb{Z} \times \mathbb{R}$. The *boundary* $\partial\Lambda_n$ is defined in the usual way as the intersection of Λ_n with the subset $\mathbb{R}^2 \setminus (-n, n)^2$ of $\mathbb{R}^2$. By (7.3), for given m, and for $\varepsilon > 0$ and sufficiently large n, the event $A_{m,n} = \{\Lambda_{m+1} \overset{-}{\leftrightarrow} \partial\Lambda_n\}^c$ satisfies $\pi_2(A_{m,n}) > 1 - \varepsilon$.

Let M_n be the subset of Λ_n containing all points connected to $\partial\Lambda_n$ by $(-)$paths of Λ_n. Thus M_n is a union of maximal intervals, and each endpoint of such an interval either lies in $\partial\Lambda_n$ (and is labelled -1), or lies in $\Lambda_n \setminus \partial\Lambda_n$ (and is labelled $+1$). Let ΔM_n be the set of all points $(u, t) \in \mathbb{Z} \times \mathbb{R}$ of $\Lambda_n \setminus M_n$ satisfying: either (i) $(u, t) \notin \partial\Lambda_n$ and (u, t) is an endpoint of a maximal interval of M_n, or (ii) there exists $e \in \{-1, +1\}$ such that $(u, t + e) \in M_n$. By the definition of M_n, every point in ΔM_n is labelled $+1$.

Let $m < n$, and let I_n be the set of all points in Λ_n reachable from Λ_m along paths of $\Lambda_n \setminus \Delta M_n$. The random set I_n is given in terms of M_n, and therefore I_n is measurable on the spin configuration of its complement $\Lambda_n \setminus I_n$. Given I_n, the spin configuration on I_n is a space–time Ising model with $+$ boundary conditions. By the FKG inequality, conditional on I_n (and the event $A_{m,n}$), the conditional π_2-measure on Λ_m is stochastically greater than π_2^+. By passing to a limit, we obtain that $\pi_2 \geq \pi_2^+$. Since $\pi_2 \leq \pi_2^+$ by elementary considerations of FKG type, we deduce that $\pi_2 = \pi_2^+$ as claimed.

One way to conclude that $M_+(2) = 0$ is to use the random-cluster representation again. By (7.2) and the above,

$$\phi_2^0(0 \leftrightarrow \infty) = \phi_2^1(0 \leftrightarrow \infty) = 0,$$

whence $M_+(2) \leq \phi_2^1(0 \leftrightarrow \infty) = 0$. □

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